Math 760 2-19-2013

Recall from last time:

A manifold is reducible if it contains a homotopically non-trivial embedded 2-sphere.

A Heegaard splitting $\underbrace{}_{M = H_1 \cup_\Sigma H_2}$ is reducible if there is an essential curve $\gamma \subset \Sigma$ s.t. γ bounds a compressing disk in both H_1 and in H_2.

Th^m (Haken)
If M is reducible then any Heegaard splitting for M is reducible.

Pf/ Last time.

Def/ A Heegaard splitting is weakly reducible if there exists essential curves $\gamma_1, \gamma_2 \subset \Sigma$ s.t. $\gamma_1 \cap \gamma_2 = \emptyset$, γ_1 bounds a compressing disk in H_1, γ_2 bounds a **compressing** disk in H_2, and Σ is not reducible.

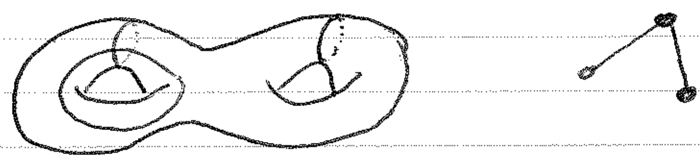
Th^m (Casson & Gordon)
If M has a weakly reducible Heegaard Splitting then M contains an embedde incompressible surface.

Def: Given a compact orientable surface F of genus ≥ 2 ~~we denote the~~ the curve complex of F, denoted $\mathcal{C}_F$, is a 1-complex with

vertices: Isotopy classes of ess. simple closed curves on F.

edges: Connect vertices with disjiont representatives.

$\boxed{\text{Ex}}$



Exercise: Show that C_F is infinite valance.

Sketch:
- Essential simple closed curves in T^2 are in 1-1 correspondence with $\mathbb{Q}$
- Similarly, there are infinitly many isotopy classes of e.s.c.c in $T^2 - \{pt\}$
- Show that for any ess s.c.c γ in F, there is a $T^2 - \{pt\}$ submanifold of F disjiont from γ.

Proposition: $\mathcal{C}_F$ is connected

Pf| Let γ_1 and γ_2 be ess. simple closed curves on F.
Isotope γ_1 and γ_2 to intersect transversely and minimally

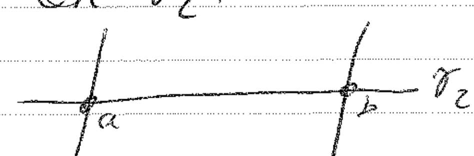
- If $|\gamma_1 \cap \gamma_2| = 0$, then γ_1 and γ_2 are connected via an edge of C_F

- If $|\gamma_1 \cap \gamma_2| = 1$, let $\eta(\gamma_i)$ be a closed annular nbh of γ_i in F. Since F is orientable $\eta(\gamma_1) \cup \eta(\gamma_2) \cong T^2$ - disk $\cong$ 

If $\partial(\eta(\gamma_1) \cup \eta(\gamma_2))$ is inessential, then it bounds a disk in F and $F \cong T^2$, a contradiction.

Hence $\gamma_1, \gamma_2 \in V(C_F)$ are connected via the edge path $[\gamma_1, \partial(\eta(\gamma_1) \cup \eta(\gamma_2))]$ $[\partial \cdots, \gamma_2]$.

If $|\gamma_1 \cap \gamma_2| = n \geq 2$, then let $a, b \in \gamma_1 \cap \gamma_2$ be consecutive points of intersection on γ_2.



Let γ_1^* and γ_1'' be the result of surgery on γ_1 along the arc between **a** and **b**.

Note that (γ_1' and γ_1'' may not be distinct)

So, one of γ_1' and γ_1'' is essential, say γ_1' ~~and mee~~, is disjiont from γ_1 and meets γ_2 is strictly fewer points.

Repeat ~~by~~ this process to construct a path from γ_1 to γ_2 in C_F. $\square$

Distance in C_F

Let $v, w \in V(C_F)$ then

$d(v, w)$ = the minimum # of edges in any path from v to w.

Lemma (Hemple)

$$d(v, w) \leq 2 + 2\log_2(i(v, w))$$

where $i(v, w)$ is the intersection number between v and w.

Distance of a bicompressible surface:

Let F ~~be a~~ $\subset M$ be a bicompressible, separating surface, of ~~distance~~ genus ≥ 2.

s.t. $M \cong H_1 \cup_F H_2$

Let V_1 = the collection of vertices of C_F bounding compressing disks in H_1

V_2 = ... in H_2

$$d(F) = d_{C_F}(V_1, V_2) = \min_{v_1 \in V_1,\, v_2 \in V_2} d_{C_F}(v_1, v_2)$$

Note

Let $M = H_1 \cup_{\Sigma} H_2$ be a Heegaard splitting.

$d(\Sigma) = 0 \iff \Sigma$ is reducible
$d(\Sigma) = 1 \iff \Sigma$ is weakly reducible.

Th^m (Hartshorn)
Let M be a closed orientable 3-manifold with H.S. $M = H_1 \cup_{\Sigma} H_2$ and

~~Then~~ Suppose M contains an orientable incompressible embedded surface F, then $d(\Sigma) \leq 2g(F)$.

* Next time basics of incompressible surfaces and the proof of Hartshorn's theorem.