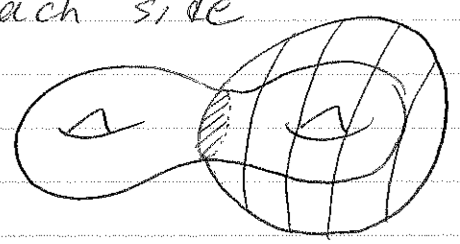
Math 760 2-14

Def: A 3-manifold M is reducible if it contains any embedded 2-sphere that does not bound a 3-ball.

Def: A Heegaard splitting (H_1, H_2, f) for a 3-manifold M is reducible if there exists ~~a cur~~ an essential curve $\gamma \subset \Sigma$ s.t. γ bounds compressing disks to each side



Thm (Haken)
Let M be a reducible manifold then any Heegaard splitting for M is reducible.

Pf| Let $M = H_1 \cup_{\Sigma} H_2$

Let S be an essential 2-sphere in M.

We can assume S is transverse to the spine of H_1.

By isotoping Σ close to the spine of H_1 we can assume $S \cap H_1$ is a collection of disks.

Let $P = S \cap H_2$, a planar surface

* • We can assume P is incompressible ... *

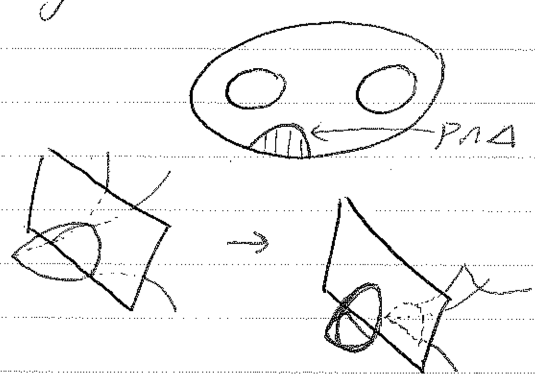
Let Δ be a <u>complete collection</u> of compressing disks for H_2

Δ

(a disjoint collection of compressing disks cutting H_2 into a single 3-ball)

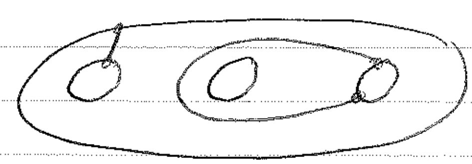
We can assume $\Delta \cap P$ contains no simple closed curves, as otherwise we can surger S along a subdisk of Δ to produce an essential 2 sphere that meets Δ in fewer curves

We can assume $\Delta \cap P$ admits no bigons in P. Since if it does, we

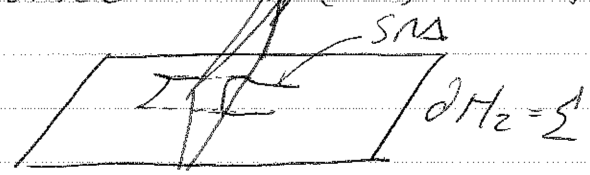


can surger Δ along this bigon to produce a complete collection of compressing disks that meet P in fewer curves

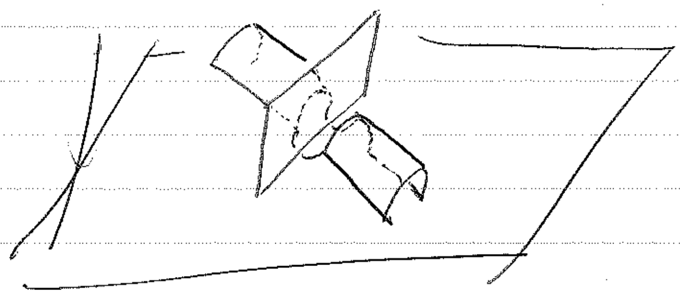
So, $P \cap \Delta$ is a collection of essential arcs in P.



Preform a sequence of isotopies of S supported in $N(\Delta)$ as follows



$\downarrow$ boundary Compressions



By preforming boundary compressions on P we convert $S \cap H_2$ into a collection of disks and $S \cap H_1$ into a planar surface.

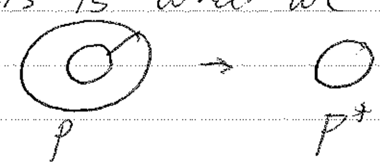
Q: What happened to $|\partial P|$

P is converted to a collection of disks P^* via boundary compressions allong a complete collection of essential arcs

Claim ~~$|\partial P| \geq |\partial P^*| - 1$~~ $|\partial P^*| \leq |\partial P| - 1$.

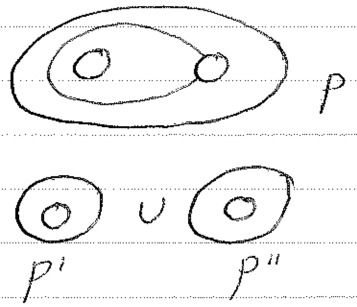
by induction on $|\partial P|$

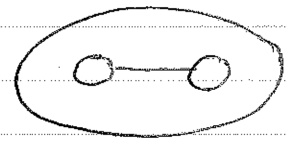
1: this is what we are trying to prove

2: 

$|\partial P| = |\partial P^*| - 1$

$|\partial P| = n$





$|\partial P|$ drops by 1 and the proof follows by induction

$|\partial P| = |\partial P'| + |\partial P''| - 1$

by induction

$\geq |\partial (P')^*| + 1 + |\partial (P'')^*| + 1 - 1$

$|\partial P| \geq |\partial P^*| + 1$

Hence $|\partial P^*| \leq |\partial P| - 1$

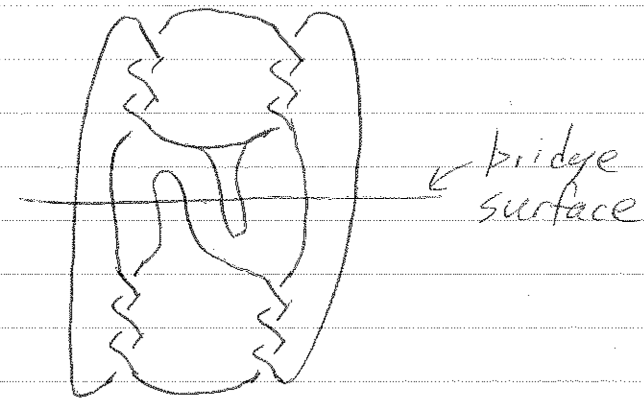
So, we have isotoped S to intersect Σ in fewer curves.

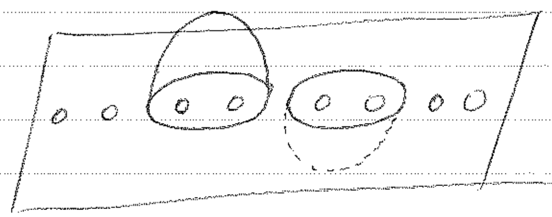
Continue this process untill we have isotoped S to meet Σ in a single curve.

Hence $H_1 \cup_{\Sigma} H_2$ is reducible. $\square$

Def: A ~~Heegaard splitting $M = H_1 \cup_{\Sigma} H_2$~~ bicompressible surface is is weakly reducible if it is not reducible and there exist ess. curves $\gamma_1, \gamma_2 \subset \Sigma$ s.t. $\gamma_1 \cap \gamma_2 = \emptyset$, γ_1 bounds a compressing disk to one side and γ_2 bounds a compressing disk to the other side.

Ex/





Th^m (Casson & Gordon)
If $M = H_1 \cup_{\Sigma} H_2$ is weakly reducible, then M contains an incompressible surface.