Math 760 Day 4 1-27-13

Thm Let M be a compact, orientable 3-manifold. Then there is a decomposition $M = P_1 \# \cdots \# P_n$ for P_i prime that is unique up to reordering and connect suming with S^3.

Pf] Existance

We can assume all 2-spheres in M are separating and M has no 2-sphere boundary components.

Let $\mathcal{T}$ be a triangulation of M.

Let S be a system of 2-spheres in M with

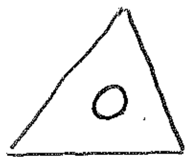
(*) No component of $M - S$ is a punctured S^3.

~~Last time: After replacing S with S' s.t. $|S'| = |S|$, S' has property (*) and S' meets every 3-simplex in disks~~

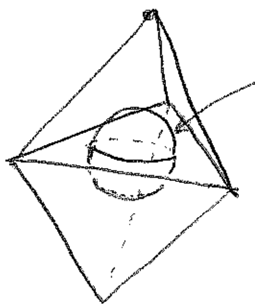
Step 1: Last time: After replacing S with S' s.t. $|S| = |S'|$ and $|S' \cap \mathcal{T}^1| \leq |S \cap \mathcal{T}^1|$, we can assume S meets every 3-simplex in a collection of disks.

Step 2: Let F be a face of $\mathcal{T}$.

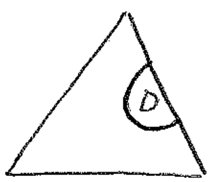
Show that we can eliminate components of $S \cap F$ of the form



- Since $S \cap \mathcal{T}$ is a collection of disks.

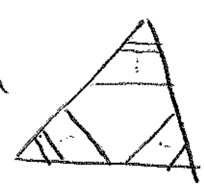


Since this 2-sphere is contained in a 3-ball it must bound a punctured S^3. $\cancel{\ast}$ to S having $(\ast)$.



- By picking an innermost such ~~curve~~ arc we can find a sub disk D of F s.t. ∂D is the endpoint union of an arc in ~~$\mathcal{T}$~~ ∂F and an arc in $S \cap F$ and $\text{int}(F) \cap S = \emptyset$.
- There is an isotopy of S supported in a nbh of D ~~and~~ that eliminates this arc and decreases $|S \cap \mathcal{T}'|$ by 2.

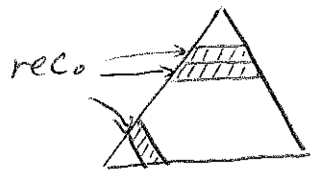
- Alternate between applying Step 1 and Step 2 until no such arcs remain.

Hence $S \cap F$ is of the form  for every F.

Step 3: Use the combinatorics of $S \cap \mathcal{T}^2$ to bound the # of interesting peices of $M-S$.

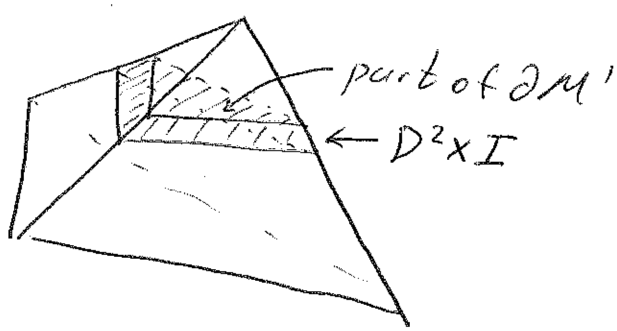
Let $\mathcal{T}$ have t faces.

There are at most $4t$ components of $\partial M-S$ that don't meet every face of $\mathcal{T}$ is rectangles



Let M' be a component of $M-S$ meeting every face of $\mathcal{T}$ in rectangles.

Then M' meets every 3-simplex in a collection of $D^2 \times I$



$M' = \bigcup_i D^2 \times I$ where the $D^2 \times I$ are glued together in a fiber preserving way along $\partial D^2 \times I$.

Hence M' is an I-bundle with 2-sphere boundary. $M' \cong S^2 \times I$ or $M' \cong \mathbb{R}P^2 \tilde{\times} I$

There are no components of $M-S$ homeo to $S^2 \times I$ by property (*).

The rank of the 2-torsion of $H_1(M)$ bounds the number of components of $M - S$ homeo to $\mathbb{R}P^2 \tilde{\times} I$.

So $M \cong P_1 \# \cdots \# P_n$ where

$n \leq$ rank of $H_1(M)$ ($S^2 \times S^1$ summands)
+ # of 2-spheres in ∂M (B^3 summands)
+ rank of 2-torsion of $H_1(M)$ ($\mathbb{R}P^3$ summands)
+ 4(# of faces in $\mathcal{T}$) (all other summands)

<u>Ex</u> Can you do better? $\square$.

Q: What is the state of the art?

Uniqueness.

Suppose $M = P_1 \# \cdots \# P_k \# l(S^1 \times S^2)$ and

$M = Q_1 \# \cdots \# Q_m \# n(S^1 \times S^2)$

where the P_i and Q_i are prime and irreducible.

Let S be a system of spheres in M s.t.

$\left(\begin{array}{l} M - S \text{ consists of } P_1, \dots, P_k \text{ with punctures} \\ \text{and punctured } S^3\text{s.} \end{array} \right)$ (+)

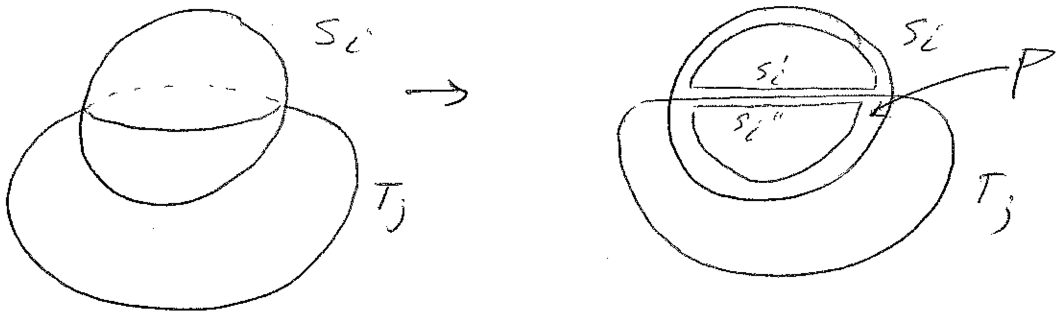
Let T be a similar system for the Q_i.

Suppose $S \cap T \neq \emptyset$.

Claim: We can ~~re-choose~~ chose S' s.t. S' has property (+) and $|S \cap T| > |S' \cap T|$.

Proof of claim.

Let γ be an innermost curve of intersection of $T \cap S$ in T



Note ~~S~~ $S \cup (S_i' \cup S_i'')$ has property $(+)$.

$S' = (S - S_i) \cup (S_i' \cup S_i'')$ has property $+$

since P is a punctured S^3. $\square$

Hence we can find a system S' with property $(+)$ and $S' \cap T = \emptyset$.

Hence $M - (S' \cup T)$ has the properties

- consists of punctured P_i and punctures S^3s.
- consists of puncture Q_i and punctured S^3s

Hence $k = m$ and, up to reordering, $P_i = Q_i$ for each i.

To show $l = n$

Note $M \cong N \# l(S^2 \times S^1) \cong N \# n(S^2 \times S^1)$

$$H_1(N) \oplus \mathbb{Z}^l \cong H_1(N) \oplus \mathbb{Z}^n$$

so $l = n$. $\square$