Lec. 9

Announcements

- H.W. Due by tomorrow morning

Outline

- Transversality of maps &
- Generalizations of the preimage Thm

Recall

Thm If $f: X \to Y$ is a smooth map and $y \in Y$ is a regular point, then $f^{-1}(y)$ is a submanifold.

Goal: Generalize the preimage theorem to give a sufficient condition for $f^{-1}(Z)$ where $Z \subset Y$ is a submanifold, to be a manifold.

Def Let $f: X \to Y$ be a smooth map and let $Z \subset Y$ be a submanifold. We say f is transversal to Z, denoted $f \pitchfork Z$, if for every $x \in f^{-1}(Z)$

$$\boxed{Im(df_x) + T_y(Z) = T_y(Y)}$$

(i.e every vector in $T_y(Y)$ can be written as a linear combination of a vector in $\mathrm{Im}(df_x)$ and ~~t_y~~ a vector in $T_y(Z)$).

<u>Thm</u>/ If $f: X \to Y$ is a smooth map and f is transversal to $Z \subset Y$, then $f^{-1}(Z)$ is a submanifold of X. Moreover, the codimension of $f^{-1}(Z)$ in X is equal to the ~~$\$$~~ codimension of Z in Y.

<u>Def</u>/ If Z is a submanifold of Y, the <u>codimension</u> of Z in Y is $\dim(Y) - \dim(Z)$.

<u>Pf</u>/ ~~Katrine~~ First, $f^{-1}(Z) \subset X \subset \mathbb{R}^N$.

Let $x \in f^{-1}(Z)$. Let $y = f(x) \in Z$.

From H.W., If Z is an l-dim'l submanifold of the k-dim'l manifold Y, then there exists a local coordinate system $\{x_1, \dots, x_k\}$ defined in a nbh U of y in Y s.t. $Z \cap U = \{v \in U \mid x_{l+1}(v) = x_{l+2}(v) = \cdots = x_k(v) = 0\}$

Recall: $x_i : U \to \mathbb{R}$ is a smooth function.

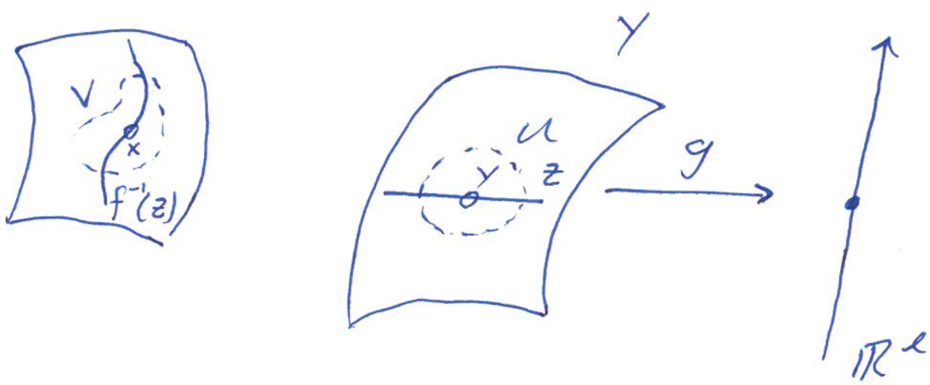
~~Pta~~

By definition, coordinate systems are Linearly independant on every point in their domain.

Hence $g : U \to \mathbb{R}^l$ given by $g(v) = (x_1(v), \dots, x_l(v))$ is a submersion on it's domain and

$g^{-1}(\vec{0}) = Z \cap U$.

Pic.



- Moreover $(g \circ f)^{-1}(\vec{0}) = f^{-1}(Z) \cap V$ for some suitable nbh V of x in X.
- We want to show $\vec{0}$ is a regular value of $g \circ f$.

Examine $d(g \circ f)_x = dg_y \circ df_x$

$d(g \circ f)_x : T_x(X) \to \mathbb{R}^l$

- $d(g \circ f)_x$ is onto iff dg_y carries $Im(df_x)$ onto $\mathbb{R}^l$.
- However dg_y is onto with kernel $T_y(Z)$.
- Hence, by linear algebra, $d(g \circ f)_x$ is onto iff $Im(df_x)$ together with $T_y(Z)$ span all of $T_y(Y)$.
- However, by def of transversal, this holds for all $x \in f^{-1}(Z)$.

Thus, $d(g \circ f)_x$ is onto for all $x \in f^{-1}(Z) \cap V$.

So, $\underbrace{(g \circ f)^{-1}(\vec{0})}^{= f^{-1}(Z) \cap V}$ is a submanifold of V,

of dimension $\dim(X) - l = \dim(X) - (\dim(Y) - \dim(Z))$.

~~So~~ It easily follows that $f^{-1}(Z)$ is a submanifold of X of ~~dimension~~ codimension the same as the codimension of Z in Y. $\square$