Lec.

Whitney embedding Th^m 1 Lec. 14

Q: Given a k-dim'l smooth manifold $X \subset \mathbb{R}^N$, what is the smallest value of N s.t. X is embedded in $\mathbb{R}^N$?

Th^m (Whitney Embedding Th^m)
If X is a smooth k-manifold, then X can be embedded in $\mathbb{R}^{2k}$.
(This is very hard to prove).

Th^m (weak Whitney Embedding)
If X is a smooth k-manifold, then X can be embedded in $\mathbb{R}^{2k+1}$.
(We will be able to prove this)

Intuition for the weak Whitney Embedding Th^m

Transversality: If X and Y are Transverse k-manifolds in $\mathbb{R}^{2k+1}$, then $X \cap Y = \emptyset$.
So, it seems resonable to expect that if we have an immersion $f: X \to \mathbb{R}^{2k+1}$ where X is a smooth k-manifold, then we can arrange for f to be "self avoiding" and elevate f to an embedding.

~~Examples~~

Recall:

preimage of every compact set is compact.

An embedding is a proper, one-to-one imersion.

Thm If $f: X \to Y$ is an embedding, the f maps. X diffeomorphically onto its image.

Examples

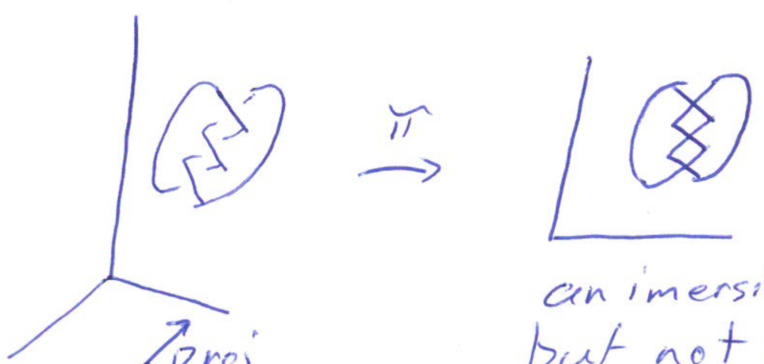
- The Whitney Embedding Thm is best possible

  Claim: There is no embedding from S^1 into $\mathbb{R}$!

  Since $\dim(S^1) = \dim(\mathbb{R}^1)$ then any immersion is also a submersion.

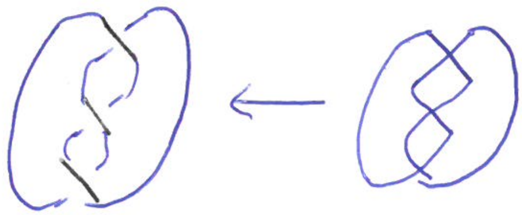
  By H.W., we know there are no submersions from compact manifolds into $\mathbb{R}^k$.

- Obviously S^1 embedds in $\mathbb{R}^2$ as the standard unit circle, but what if we embedded S^1 in $\mathbb{R}^3$ and tried to deduce dimensions via projection maps.

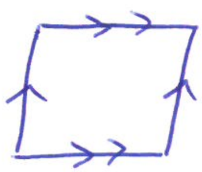


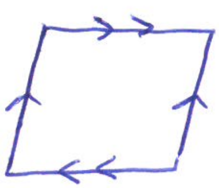
an imersion of S^1 into $\mathbb{R}^2$
but not an embedding

How to turn an emersion in $\mathbb{R}^k$ into an embedding in $\mathbb{R}^{k+1}$



use color to vissualize change in an orthogonal dirrection

 $\cong$  $S^1 \times S^1$ = torus

  $S^1 \tilde{\times} S^1$ = klein bottle.

Suppose X is a smooth manifold in $\mathbb{R}^N$

The <u>tangent bundle</u> of X in $\mathbb{R}^N$ is defined by $T(X) = \{(x,v) \in X \times \mathbb{R}^N ; v \in T_x(X)\}$

(note: $T(X)$ is <u>not</u> generally diffeomorphic to $X \times \mathbb{R}^k$ where $\dim(X) = k$.)

(i.e. the tangent bundle for the standard copy of S^2 in $\mathbb{R}^3$ is <u>not</u> $S^2 \times \mathbb{R}^2$ by hairy ball thm).

* Note $T(X) \subset \mathbb{R}^N \times \mathbb{R}^N$.

* Given $f: X \to Y$ a smooth map
  $df: T(X) \to T(Y)$ defined by
  $df(x,v) = (f(x), df_x(v))$
  is a smooth map.

* If $f: X \to Y$ and $g: Y \to Z$ are smooth
  then $d(g \circ f) = dg \circ df$ by the chain rule.

* If $f: X \to Y$ is a diffeomorphism, then
  $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$
  $df^{-1} \circ df = id_{T(X)}$ and $df \circ df^{-1} = id_{T(Y)}$

Let $X \subset \mathbb{R}^N$ be a k-manifold.

Let $\phi: U \to W$ be a local parameterization of an open set $W \subset X$.

Since ϕ is a diffeomorphism, then

$$d\phi: T(U) \to T(W) \text{ is a diffeo.}$$

$T(U) = U \times \mathbb{R}^k \cong \mathbb{R}^{2k}$

$T(W) = T(X) \cap (W \times \mathbb{R}^N)$, so $T(W)$ is an open subset of $T(X)$.

Thus $d\phi$ is a local parametrization of $T(X)$.

Conclusion: $T(X)$ is a manifold and $\dim(T(X)) = 2\dim(X)$.

<u>Theorem</u> Every k-dimensional smooth manifold admitts a one-to-one imersion into $\mathbb{R}^{2k+1}$.

<u>Pf</u> Let $X \subset \mathbb{R}^N$ be a smooth k-manifold.

Suppose $N > 2k+1$.

Define $h: X \times X \times \mathbb{R} \to \mathbb{R}^N$ by $h(x, y, t) = t(f(x) - f(y))$

Define $g: T(X) \to \mathbb{R}^N$ by $g(x, v) = df_x(v)$

Claim: If $f: X \to Y$ is a smooth map and $\dim(X) < \dim(Y)$, then $f(X)$ has measure zero in Y.

<u>Pf</u> By the preimage Thm, f has <u>no</u> regular values. By Sards Thm, $f(X)$ has measure zero. $\square$

By the claim, both $Im(h)$ and $Im(g)$ have measure zero in $\mathbb{R}^N$. Hence we can find a non-zero vector $a \in \mathbb{R}^N$ s.t. $a \notin Im(h)$ and $a \notin Im(g)$.

Define $\pi : \mathbb{R}^N \to H$ be the projection map where H is the $(N-1)$-dimensional subspace of $\mathbb{R}^N$ that is orthoganal to a.

Claim: $\pi \circ f$ is injective.

Pf| Suppose $\pi(f(x)) = \pi(f(y))$, then $\exists\, t \in \mathbb{R}$ s.t.

$$f(x) - f(y) = t \cdot a$$

$$1/t\,(f(x) - f(y)) = a$$

which contradicts $a \notin Im(h)$ unless $x = y$. $\square$

Claim: $\pi \circ f$ is an imersion.

Pf| Suppose $\exists\, x \in X$ and $v \in T_x(X)$ s.t. $\vec{v} \neq 0$ s.t.

$$d(\pi \circ f)_x(v) = 0.$$

$$d\pi_{f(x)} \circ df_x(v) = 0$$

$$\pi \circ df_x(v) = 0$$

Thus $df_x(v) = t \cdot a$ for $t \in \mathbb{R} \setminus \{0\}$

$df_x(1/t\, v) = a$ for $t \in \mathbb{R} \setminus \{0$

So $g(x, 1/t\, v) = a$ a contradiction to our choice of a. $\square$

Hence, $\pi \circ f$ is an injective immersion into $\mathbb{R}^{N-1}$.

By induction, there exists an injective immersion from X into $\mathbb{R}^{2k+1}$. $\square$