Lec. 10

Announcements

- HW 3 posted to the web.

From last time

Def Given a smooth map $f: X \to Y$ and a submanifold $Z \subset Y$, f is <u>transversal</u> to Z, denoted $f \pitchfork Z$ if for every $x \in f^{-1}(Z)$

$$\text{Image}(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y).$$

Thm If $f \pitchfork Z$, then $f^{-1}(Z)$ is a submanifold of X and $\text{codim}_X(f^{-1}(Z)) = \text{codim}_Y(Z)$.

Important Case

X and Z are submanifolds of Y and $f: X \to Y$ is the inclusion map.

$$f^{-1}(Z) = X \cap Y$$

and $\overset{Y \pitchfork Z}{f \pitchfork Z} \iff T_x(X) + T_x(Z) = T_x(Y) \ \forall x \in X \cap Y.$

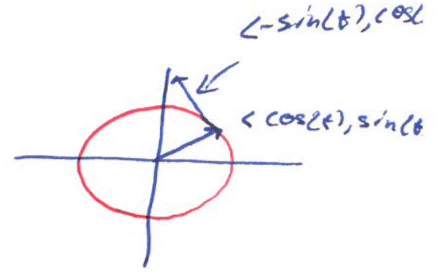
Ex $Y = \mathbb{R}^3$, $X = xy\text{-plane}$, $Z = S^2$

$$f(x,y) = (x,y,0)$$

To show $f \pitchfork Z$, we will show $T_x(X) + T_x(Z) = T_x(Y)\ \forall x \in X \cap Y$

Let $(x,y,z) \in \mathbb{R}^2 \cap S^2$, so, $(x,y,z) = (\cos(t), \sin(t), 0)$ for some $t \in [0, 2\pi]$.

$$T_{(\cos(t),\, \sin(t),\, 0)}(\mathbb{R}^3) = \mathbb{R}^3$$



$$T_{(\cos(t),\, \sin(t),\, 0)}(xy\text{-plane}) = xy\text{-plane}$$

$$T_{(\cos(t),\, \sin(t),\, 0)}(S^2) = \text{span}(\langle 0,0,1\rangle, \langle -\sin(t), \cos(t), 0\rangle)$$

$$T_{(\cos(t),\, \sin(t),\, 0)}(xy\text{-plane}) + T_{(\cos(t),\, \sin(t),\, 0)}(S^2)$$

$$\supset \text{span}((1,0,0),\ (0,\phi,0),\ (0,0,1))$$

Hence, $f \pitchfork S^2$.

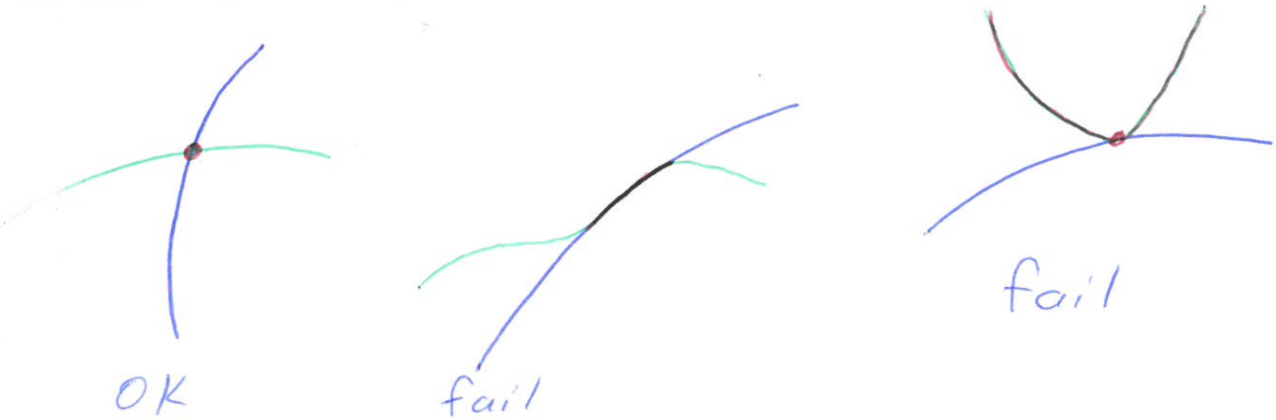
So, $S^2 \cap \mathbb{R}^2$ is a submanifold of dimension 1.

Def| If X and Z are submanifolds of Y and $T_x(X) + T_x(Z) = T_x(Y)\ \forall x \in X \cap Z$, then we say X and Z are transversal.

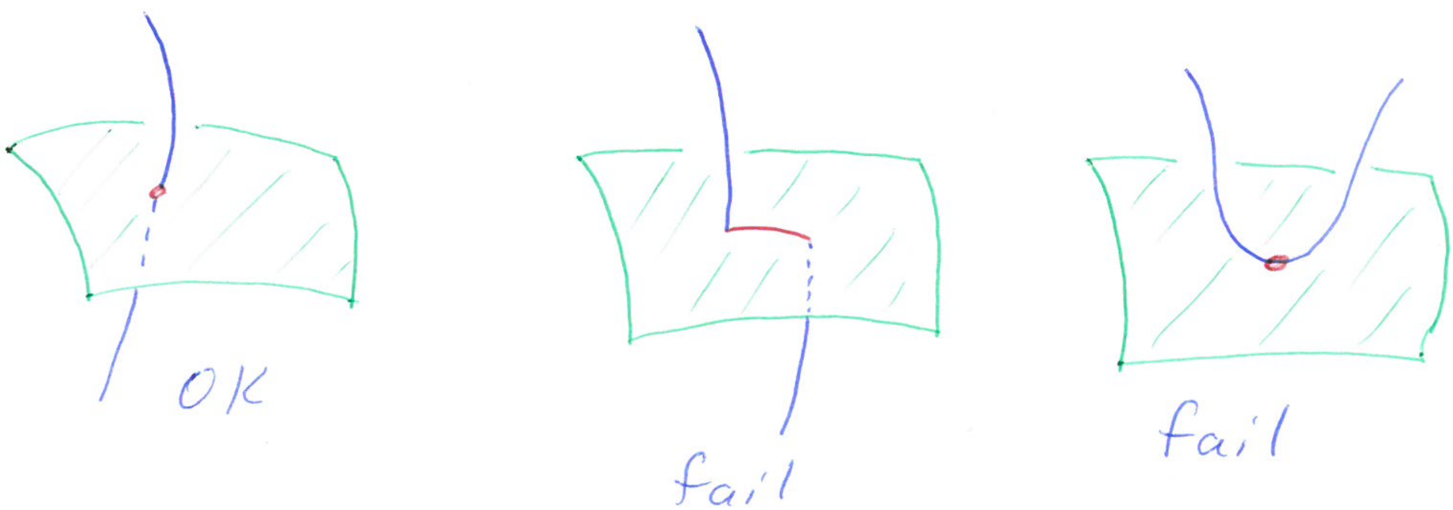
Cor| If X and Z are transversal submanifolds of Y, Then $X \cap Z$ is a submanifold and $\text{codim}_Y(X \cap Z) = \text{codim}_Y(X) + \text{codim}_Y(Z)$.

Q: How do submanifolds fail to be transversal.

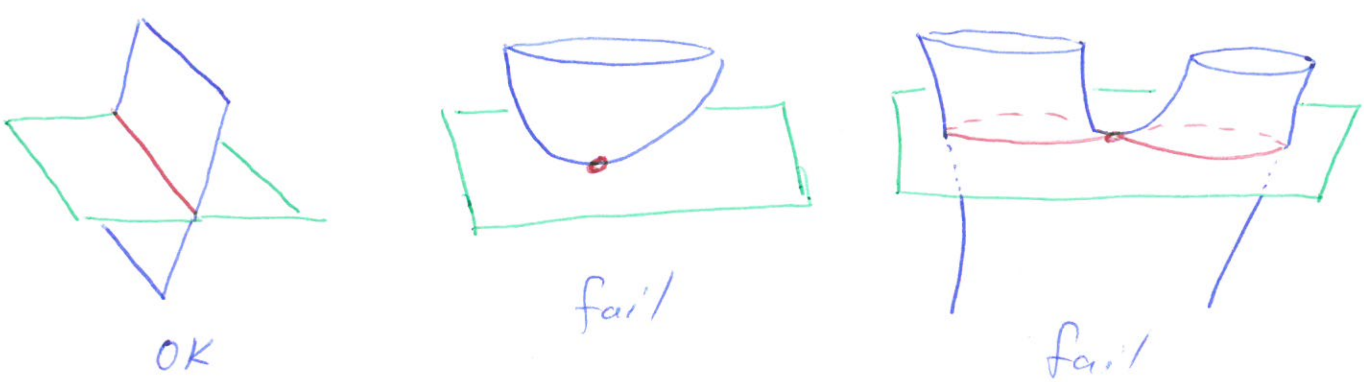
Curves in $\mathbb{R}^2$



Curves and surfaces in $\mathbb{R}^3$.



Surfaces in $\mathbb{R}^3$



Def Let $f_0 : X \to Y$ and $f_1 : X \to Y$ be smooth maps, we say f_0 is <u>homotopic</u> to f_1 if there exists a smooth map $F : X \times [0,1] \longrightarrow Y$ s.t. $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

Notation: If f_0 is homotopic to f_1 we denote it by $f_0 \sim f_1$.

Exercise: $\sim$ induces an equivalence relation on the set of smooth maps from X to Y.

Ex The identity map on $\mathbb{R}^k$ is homotopic to the constant map.

$$f_0(\vec{x}) = \vec{x}$$

$$f_1(\vec{x}) = \vec{0}$$

$$F : \mathbb{R}^k \times [0,1] \longrightarrow \mathbb{R}^k \text{ by } F(\vec{x}, t) = (1-t)\vec{x}$$

(F is smooth since each component function is a polynomial).

Def A property of smooth maps is <u>stable</u> provided that whenever $f_0 : X \to Y$ possesses the property and $f_t : X \to Y$ is a homotopy of f_0, then, for some $\varepsilon > 0$, each f_t with $t < \varepsilon$ also possesses this property.

Stability Thm The following classes of smooth maps from a compact manifold X into a manifold Y are stable classes.

a) local diffeomorphism

b) immersions

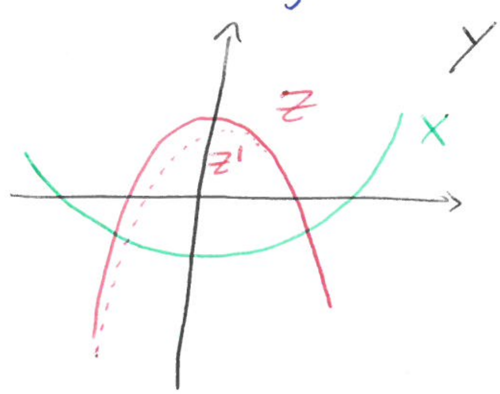
c) submersions

d) maps transversal to a specified submanifold $Z \subset Y$.

e) embeddings

f) diffeomorphisms.

Idea: small smooth changes do not effect "generic" smooth conditions



$X \pitchfork Z$

$X \pitchfork Z'$

What about