Announcements

6–7 Fri
3:30–5 Monday

- Midterm in class on Tuesday
  Covers §51, 52, 53, 54 Munkres
  Homeworks 1, 2, 3.
- New HW 4 posted and due a week from today.
- Extra Office Hours Tomorrow and Monday.

Outline

- $\pi_1(S^1,(1,0)) \cong \mathbb{Z}$
- Brouwer fixed pt theorem.

Def| If $A \subset X$, a <u>retraction</u> of X onto A is a continuous map $r: X \to A$ s.t. $r|_A = id_A$. If such a map exists, we say A is a <u>retract</u> of X.

Lemma 55.1| If A is a retract of X, then the homomorphism on fundamental groups induced by the inclusion map is one-to-one.

Pf| Let $j: A \to X$ via $j(a) = a$ be the inclution map.
Then $j_*: \pi_1(A, a_0) \to \pi_1(X, a_0)$ is the induced map.
Let $r: X \to A$ be the retration of X onto A.
Then $r_*: \pi_1(X, a_0) \to \pi_1(A, a_0)$ is the induced map.

Note $r \circ j : A \to A$ is the identity map.

Hence, by ~~52~~5?.4, $(r \circ j)_* = id_{\pi_1(A, a_0)}$.

~~So $r_* \circ j_*$~~ by 52.4 $(r \circ j)_* = r_* \circ j_*$.

Thus, $r_* \circ j_* = id_*$.

Hence j_* is one-to-one and

r_* is onto. $\square$

Th^m | 55.2 There is no retraction of B^2 onto S^1.

Pf | Recall, since B^2 is convex, $\pi_1(B^2, x_0) \cong \{1\}$

Also, $\pi_1(S^1, x_0) \cong \mathbb{Z}$. Suppose, to form a contradiction, that there is a retraction $r : B^2 \to S^1$. By theorem 55.1, the inclution $j : S^1 \to B^2$ induces a one-to-one map $j_* : \pi_1(S_1, x_0) \to \pi_1(B^2, x_0)$

$$j_* : \mathbb{Z} \to \{1\}.$$

However, there are no one-to-one maps between $\mathbb{Z}$ and $\{1\}$. $\divideontimes$

Thus, no retraction can exist. $\square$

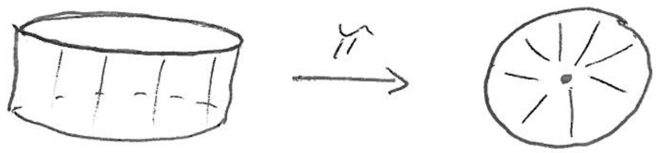
Lemma 55.3 Let $h: S^1 \to X$ be a continuous map.
Then the following are equivalent.

1) h is nulhomotopic
2) h extends to a continuous map $k: B^2 \to X$
3) h_* is the trivial homomorphism

Pf 1) $\Rightarrow$ 2)

Since h is null homotopic, let $H: S^1 \times I \to X$ be a homotopy taking h to a constant map e_{x_0}.

Let $\pi: S^1 \times I \to B^2$ be the map
$$\pi(x,t) = (1-t)x.$$
One can check that this is a quotient map.



Define $k: B^2 \to X$ by $k(z) = \begin{cases} H(\pi^{-1}(z)) & z \neq \vec{0} \\ H((x_0, 1)) & z = \vec{0} \end{cases}$

One can check that this map is continuous.

Note that $k\big|_{\partial B^2}(x) = H(x,1) = h(x)$

So, k is a continuous extention of h to B^2.

(2) $\Rightarrow$ (3) If $j: S^1 \to B^2$ is the inclusion map, then $h = k \circ j$. Hence, $h_* = k_* \circ j_*$.

But, $j_* : \pi_1(S^1, b_0) \to \pi_1(B^2, b_0)$

is trivial since $\pi_1(B^2, b_0) \cong \{1\}$, Hence h_* must be trivial.

(3) $\Rightarrow$ (1) Let $p: \mathbb{R} \to S^1$ be the standard covering map, and let $p|_I$ is a loop in S^1 based at $(1,0)$.

From the proof of Thm 54.5, $[p|_I]$ generates $\pi_1(S^1, (1,0))$.

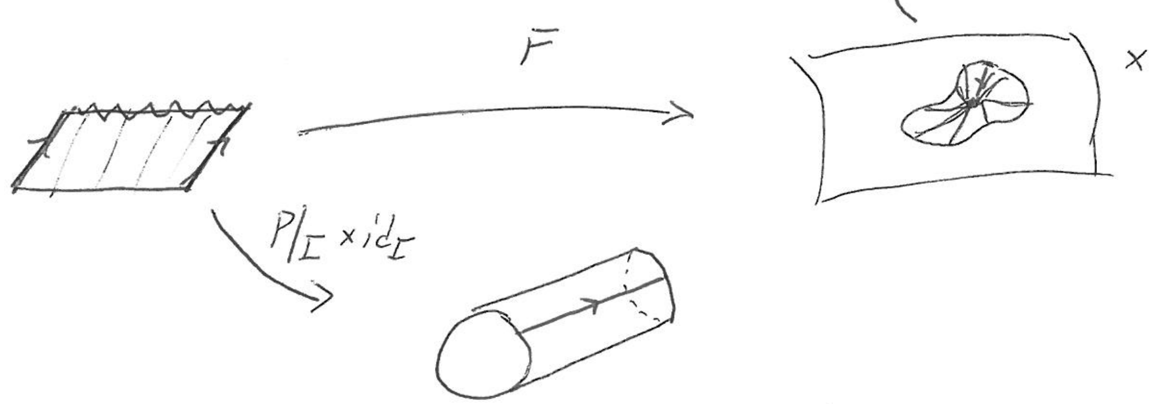
Let $x_0 = h((1,0))$. Since h_* is trivial,

$$[h \circ p|_I] = [e_{x_0}] \in \pi_1(X, x_0)$$

Let $F: I \times I \to X$ be the path homotopy from

$h \circ p|_I$ to e_{x_0}

define $H: S^1 \times I \to X$ by $H(x,t) = \begin{cases} F \circ (p|_I \times id_I)^{-1}(x,t) \\ \end{cases}$



Can check $H(x,t)$ is well defined. and continuous

$$H(x,1) = F \circ (P|_I \times id_I)^{-1}(x,1)$$
$$= F(x,1)$$
$$= h(x)$$
$$H(x,0) = F \circ (P|_I \times id_I)^{-1}(x,0)$$
$$= F(x,0)$$
$$= e_{x_0}$$

Hence, $h \simeq_p e_{x_0}$

So, h is null-homotopic.