Announcements

- Midterm a week from today
  Covers Sections 51, 52, 53, 54 of Munkres
  and homeworks 1, 2, 3.
- New Homework due Thursday of next week

Outline

- Review
- $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$

Review

Lemma (54.1) Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$. Any path $f: I \to B$ s.t. $f(0) = b_0$ has a unique lift $\check{f}: I \to E$ s.t. $\check{f}(0) = e_0$.

Lemma (54.2) Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$. Any path homotopy $F: I \times I \to B$ s.t. $F(0,0) = b_0$ has a unique lift $\check{F}: I \times I \to E$ s.t $\check{F}(0,0) = e_0$.

Lemma (54.3) Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$. If $f, g: I \to B$ are two paths from b_0 to b_1 and $\check{f}, \check{g}: I \to E$ are the unique lifts starting at e_0, then if $f \simeq_p g$, then $\check{f} \simeq_p \check{g}$ and $\check{f}(1) = \check{g}(1)$.

Pf| Let $F: I\times I \to B$ be the path homotopy between f and g s.t. $F(0,0) = b_0$.

By lemma 54.2, there is a unique lift of F to a path homotopy $\tilde{F}: I\times I \to E$ ~~between~~ ~~$\tilde{f}$ and $\tilde{g}$ s.t. $\tilde{F}(0,0) = e_0$. Hence~~ s.t. $\tilde{F}(\{0\}\times I) = e_0$ and $\tilde{F}(\{1\}\times I) = e_1$.

More over $\tilde{F}|_{I\times\{0\}}$ is a lift of f that begins at e_0.

By Lemma 54.1, $\tilde{F}|_{I\times\{0\}}(s,0) = \tilde{f}(s)$.

Similarly, $\tilde{F}|_{I\times\{1\}}(s,1) = \tilde{g}(s)$.

Hence $\tilde{F}$ is a path homotopy between $\tilde{f}$ and $\tilde{g}$ s.t. $\tilde{f}(0) = \tilde{g}(0) = e_1$. $\square$

Def| Let $p: E \to B$ be a covering map s.t. $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the unique lift of f that begins at e_0. The lifting correspondence is the map

$$\phi: \pi_1(B, b_0) \to p^{-1}(b_0)$$

by $\phi([f]) = \tilde{f}(1)$.

Note:- By Lemma 54.3, ϕ is well defined.

- ϕ is dependent on the choice of e_0

Thm (54.4) Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$. If E is path-connected the lifting correspondence is onto. If E is simply connected, the lifting correspenence is bijective.

Pf| Suppose E is path connected.

WTS $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective when $\phi([f]) = \tilde{f}(1)$.

Let $e_* \in p^{-1}(b_0)$. Since E is path connected, there exists a path $g: I \to E$ s.t. $g(0) = e_0$ and $g(1) = e_*$. $p \circ g: I \to B$ is a loop based at b_0. Hence $\phi([p \circ g]) = g(1) = e_*$. Thus, ϕ is onto.

Suppose E is simply connected.

Hence, E is path-connected and $\pi_1(E, e_0) \cong \{1\}$.

Let $[f], [g] \in \pi_1(B, b_0)$ s.t. $\phi([f]) = \phi([g])$.

Thus $\tilde{f}(1) = \tilde{g}(1)$.

Since $\tilde{f}$ and $\tilde{g}$ both ~~being~~ begin and end at the same points, ~~$\bar{\tilde{g}} * \tilde{f}$ is a loop in E based at e_0~~. and $\pi_1(E, e_0) \cong \{1\}$, then there is a path homotopy $\tilde{F}$ from $\tilde{f}$ to $\tilde{g}$. By 52.3 Then $p \circ \tilde{F}$ is a path homotopy between f and g. Thus $[f] = [g]$.

So, ϕ is a bijection.

$\boxed{\text{Thm (54.5)}}$ $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$.

$\boxed{\text{Pf}}$ Let $p: \mathbb{R} \to S^1$ be the covering map given by $p(x) = (\cos(2\pi x), \sin(2\pi x))$, $p(0) = (1,0)$.

By Thm 54.4, $\phi: \pi_1(S^1, (1,0)) \to p^{-1}((1,0)) = \mathbb{Z}$ is a bijection.

<u>Claim</u> ϕ is a homomorphism.

Let $[f], [g] \in \pi_1(S^1, (1,0))$ s.t.

$\tilde{f}(1) = n \in p^{-1}((1,0)) = \mathbb{Z}$ and

$\tilde{g}(1) = m \in p^{-1}((1,0)) = \mathbb{Z}$.

Hence $\phi([f]) = n$ and $\phi([g]) = m$.

Define $\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$ a path in $\mathbb{R}$ from n to $n+m$.

Note that $p \circ \tilde{\tilde{g}}(s) = p(n + \tilde{g}(s)) = p(\tilde{g}(s)) = g(s)$.

Hence $\tilde{\tilde{g}}(s)$ is a lift of g.

Hence, $\tilde{f} * \tilde{\tilde{g}} : I \to \mathbb{R}$ is a path from 0 to $n+m$.

$$p \circ (\tilde{f} * \tilde{\tilde{g}}) = p \circ \tilde{f} * p \circ \tilde{\tilde{g}} = f * g.$$

So, $\tilde{f} * \tilde{\tilde{g}}$ is a lift of $f * g$ beginning at 0.

Thus $\phi([f*g]) = \tilde{f} * \tilde{\tilde{g}}(1) = n + m = \phi([f]) + \phi([g])$

$\square$