Topology Day 8

Outline

- Pasting Lemma
- Topologies on Products

Announcements

- Exam a week from today

Pasting Lemma Let X be a top space and A, B be closed sets in X s.t. $X = A \cup B$. Suppose $f: A \to \cancel{B}$ and $g: B \to Y$ are continuous maps s.t. $f(x) = g(x)$ for all $x \in A \cap B$. Then

$$h: X \to Y \quad \text{s.t. } h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases} \quad \text{is continuous.}$$

Pf Let ~~U be an~~ C be a closed set in Y.

$$\begin{aligned} h^{-1}(C) &= \{x \in X \mid h(x) \in C\} \\ &= \{x \in A \mid f(x) \in C\} \cup \{x \in B \mid g(x) \in C\} \\ &= f^{-1}(C) \cup g^{-1}(C) \end{aligned}$$

By continuity of f and g, $f^{-1}(C)$ is closed <u>in A</u> and $g^{-1}(C)$ is closed <u>in B</u>.

By Lemma 17.2 in Munkres,

$f^{-1}(C)$ is closed in A iff $\exists\ K_A \subset X$ a closed set s.t. $f^{-1}(C) = K_A \cap A$.

Similarly $g^{-1}(C) = K_B \cap B$ for K_B closed in X.

Hence $h^{-1}(C) = (K_A \cap A) \cup (K_B \cap B)$.

Since finite intersections and finite unions of closed is closed, then $h^{-1}(C)$ is closed.

Hence h is continuous. $\square$

Box and Product topologies

Recall: Given top. spaces (X, τ_X) and (Y, τ_Y) the topology on $X \times Y$ is generated by basis $\{U \times V \mid U \in \tau_X, V \in \tau_Y\}$.

Motivating Question: what topologies make sence on infinite products $\prod_{\alpha \in J} X_\alpha$

$$\prod_{\alpha \in J} X_\alpha = \left\{ \vec{X} : J \to \bigcup_{\alpha \in J} X_\alpha \;\middle|\; \vec{X}(\alpha) \in X_\alpha \right\}$$

$$\vec{X} \in \prod_{i=1}^{\infty} \mathbb{R}, \quad \vec{X} = \{X_n\}_{n=1}^{\infty} \text{ or } \vec{X} : \mathbb{Z}^+ \to \mathbb{R}$$

Def Let $\{X_\alpha\}_{\alpha\in J}$ be a collection of top. spaces.

The <u>box topology</u> on $\prod_{\alpha\in J} X_\alpha$ has basis

$$\mathcal{B}_{box} = \left\{ \prod_{\alpha\in J} U_\alpha \,\middle|\, U_\alpha \text{ is open in } X_\alpha \right\}$$

Exercise: Check this is a basis.

Def The <u>product topology</u> on $\prod_{\alpha\in J} X_\alpha$ has basis

$$\mathcal{B}_{prod} = \left\{ \prod_{\alpha\in J} U_\alpha \,\middle|\, \begin{array}{l} U_\alpha \text{ is open in } X_\alpha \text{ and} \\ U_\alpha = X_\alpha \text{ for all but finitly} \\ \text{many } \alpha\in J \end{array} \right\}$$

Exercise: Check that this is a basis.

Remarks

1. $\mathcal{B}_{box} \supset \mathcal{B}_{prod}$ so box topology is finer than the product topology
2. For finite products the Box and Product topologies are the same
3. Alternate interpritation of the product top: Topology with sub-basis

$$S = \{ \pi_\alpha^{-1}(U_\alpha) : \alpha\in J \text{ and } U_\alpha \subset X_\alpha \text{ is open} \}$$

where $\pi_\alpha : \prod_{\alpha\in J} X_\alpha \to X_\alpha$ is the projection.

Why? Finite intersections $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$ are exactly the elements of $\mathcal{B}_{prod}$.

Facts If X_α has basis $\mathcal{B}_\alpha$ for each $\alpha \in J$, then $\{\prod_{\alpha\in J} B_\alpha \mid B_\alpha \in \mathcal{B}_\alpha\}$ is a basis for the box topology.

Similarly $\left\{\prod_{\alpha\in J} B_\alpha \;\middle|\; \begin{array}{l} B_\alpha \in \mathcal{B} \text{ for finitly many } \alpha \\ \text{and } B_\alpha = X_\alpha \text{ otherwise} \end{array}\right\}$ is a basis for the product topology.

Ex $\prod_{i=1}^{\infty} (0,1) \subset \prod_{i=1}^{\infty} \mathbb{R}$ is open in the box topology but is <u>not</u> open in the product topology.

Prop If $\{X_\alpha\}_{\alpha\in J}$ is a collection of Hausdorff spaces, then $\prod_{\alpha\in J} X_\alpha$ is Hausdorff in both the prod. and box topologies

Pf Exercise.

Prop Let $A_\alpha \subset X_\alpha$ for each $\alpha \in J$. Let $\prod_{\alpha\in J} A_\alpha$ have the product (box) topology. Then the product (box) topology on $\prod A_\alpha$ agrees with the subspace topology $\prod A_\alpha \subset \prod X_\alpha$.

Prop| If $A_\alpha \subseteq X_\alpha$, then in either the box or product topology, $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

The following is one reason why we prefer the product topology

Prop) Let $f: Y \to \prod_{\gamma \in J} X_\gamma$ where $\prod X_\alpha$ has the product top. Then f is cont. iff each $f_\beta = \pi_\beta \circ f : Y \to X_\beta$ is cont.

Pf| $\Rightarrow$| Let $f_\beta = \pi_\beta \circ f$. Note that π_β is cont. (as proved last time) and f is continuous by assumption. Hence f_β is continuous since the composition of continuous maps is continuous.

$\Leftarrow$| Suppose each f_β is cont. We will show that f^{-1}(any sub-basis element) is open. This implies f is continuous. Let $\pi_\beta^{-1}(U_\beta) \in S$ a sub basis for the product top.

Open in X_β

$$f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \circ f)^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$$

Hence $f^{-1}(\pi_\beta^{-1}(U_\beta))$ is open by continuity of f_β. $\square$

$\Leftarrow$ Is false for the ~~product~~ box topology

Ex] $\prod_{i=1}^{\infty} \mathbb{R}$ with the box topology.

Let $f: \mathbb{R} \to \prod_{i=1}^{\infty} \mathbb{R}$ be $f(t) = (t, t, \dots)$

$f_i : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x$ is continuous for each i

However $B = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \cdots$ is open in $\prod_{i=1}^{\infty} \mathbb{R}$.

$f^{-1}(B) = \{ x \in \mathbb{R} \mid \text{s.t. } x \in (\frac{-1}{n}, \frac{1}{n}) \text{ for } \underline{\text{every}}\ n \in \mathbb{Z}^{+} \}$

$= \{0\}$ ← not open.