Topology Day 7

- Outline
- Continuity

25	27	4
6	8	9

- Vote on midterm date. Tue Feb 25, 27 or March 4th

Recall: If X and Y are top. spaces, $f: X \to Y$ is continuous iff $f^{-1}(U)$ is open in X whenever U is open in Y.

First we verify that this def. is compatable with the standard def of continuity on $\mathbb{R}$.

<u>Prop</u> | Let $f: \mathbb{R}_s \to \mathbb{R}_s$. Then f is continuous <u>iff</u> given any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

<u>Pf</u> | <u>$\Rightarrow$</u> | Suppose f is continuous and let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ be given.

Since $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ is open and f is continuous, $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ is open <u>and</u> contains x_0.

By def. of open w.r.t. a basis $\exists\, (a,b)$ s.t.

$$x_0 \in (a,b) \subset f^{-1}(\;).$$

WLOG we can pick (a,b) symetric about x_0.

$\exists\, \delta$ s.t. $x_0 \in (x_0 - \delta, x_0 + \delta) \subset f^{-1}((\;))$

There exist $\delta > 0$ s.t. if $|x_0 - x| < \delta$ then $|f(x_0) - f(x)| < \varepsilon$.

$\Leftarrow$ | Exercise

Remark | When verifying continuity, it is enough to check that the inverse image of basis elements (or even subbasis elements) are open.

~~Prop | Let $f: X \to Y$ have the property that~~

Prop | Let X be a top. space and Y be a top. space with basis $\mathcal{B}$. If $f: X \to Y$ has the property that $f^{-1}(B)$ is open for every $B \in \mathcal{B}$, then f is continuous

Pf | Let U be an open set in Y.

$$U = \bigcup_{\alpha \in J} B_\alpha \text{ for } B_\alpha \in \mathcal{B}.$$

$$f^{-1}\Big(\bigcup_{\alpha \in J} B_\alpha\Big) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha).$$

Since $f^{-1}(B_\alpha)$ is open and the union of open is open, then $f^{-1}\big(\bigcup_{\alpha \in J} B_\alpha\big)$ is open. $\square$

Equivalent formulations of continuity

Thm | Let $f: X \to Y$ be a map between top spaces

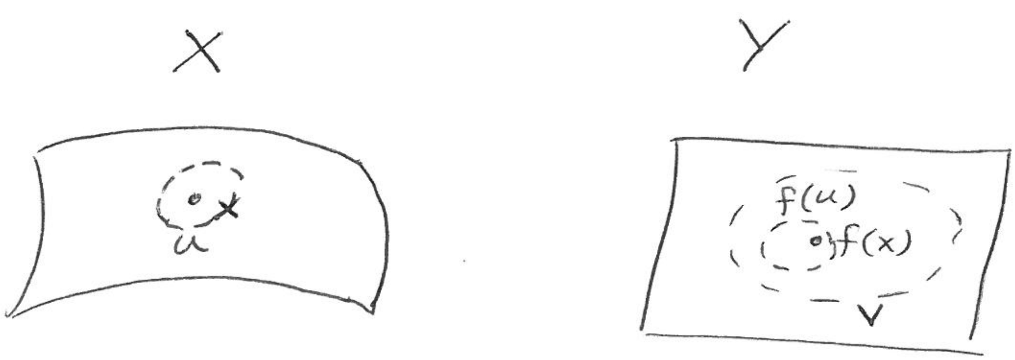
TFAE:

1) f is continuous
2) $\forall A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$
3) $\forall B \subset Y$ s.t. B is closed, $f^{-1}(B)$ is closed.
4) $\forall x \in X$ and all nbhs V of $f(x)$, $\exists$ a nbh U of x with $f(U) \subset V$.

Inturpritations

2) f sends every set and its limit points to $f(A)$ and its limit points.

4)



<u>Pf</u> We will show $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 1)$ and $1) \Leftrightarrow 4)$.

<u>$1) \Rightarrow 2)$</u> Suppose f is continuous.

Let $A \subset X$.

Want to show if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$.

Let $x \in \overline{A}$. Let V be a nbh of $f(x)$.

By continuity $f^{-1}(V)$ is a nbh of x.

Since $x \in \overline{A}$, $f^{-1}(V) \cap A \neq \emptyset$.

Let $y \in f^{-1}(V) \cap A$. $f(y) \in V \cap f(A) = f(f^{-1}(V) \cap A)$.

Since V was arbitrary, $f(x) \in \overline{f(A)}$. $\square$

<u>$2) \Rightarrow 3)$</u> Let B be a closed subset of Y.

Let $A = f^{-1}(B)$. W.T.S. A is closed in X.

Let $x \in \overline{A}$. By 2), $f(x) \in \overline{f(A)} = \overline{f(f^{-1}(B))}$.

Since $f(f^{-1}(B)) \subset B$, $f(x) \in \overline{B} = B$. Hence, $x \in A$. Since, $\overline{A} \subset A$ then A is closed.

Thus $f^{-1}(B)$ is closed. $\square$

$3) \Rightarrow 1)$ Let $\mathcal{U} \subset Y$ be open.

$Y - \mathcal{U}$ is closed.

By 3), $f^{-1}(Y-\mathcal{U})$ is closed.

Hence, $X - f^{-1}(Y-\mathcal{U})$ is open.

Fact: $f^{-1}(Y-\mathcal{U}) = X - f^{-1}(\mathcal{U})$.

So, $X - (X - f^{-1}(\mathcal{U})) = f^{-1}(\mathcal{U})$ is open.

$1) \Rightarrow 4)$ Let $x \in X$ and let V be a nbh of $f(x)$.

By 1), $f^{-1}(V)$ is ~~open in~~ a nbh of x in X.

Since $f(f^{-1}(V)) \subset V$, then we satisfy 4). $\square$

$4) \Rightarrow 1)$ Let $\mathcal{U}$ be open in Y.

By 4), $\forall x \in f^{-1}(\mathcal{U})$ $\exists \mathcal{U}_x$ a nbh of x s.t. $f(\mathcal{U}_x) \subset V$.

Clearly $f^{-1}(\mathcal{U}) \subset \bigcup_{x \in f^{-1}(\mathcal{U})} \mathcal{U}_x$.

Since $\mathcal{U}_x \subset f^{-1}(f(\mathcal{U}_x)) \subset f^{-1}(\mathcal{U})$

then $\bigcup_{x \in f^{-1}(\mathcal{U})} \mathcal{U}_x \subset f^{-1}(\mathcal{U})$

Thus, $f^{-1}(\mathcal{U})$ is the union of open sets and is itself open $\square$.

Question | What does it mean for two top. spaces to be equivalent?

Def | Let X and Y be top. spaces. $f: X \to Y$ is a homeomorphism if f is continuous, f is a bijection and f^{-1} is continuous.

X is equivalent to Y if there exists a homeomorphism from X to Y.

Ex | $\mathbb{R}$ is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$f: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ s.t. $f(x) = \tan^{-1}(x)$
(to verify continuity use ε-δ def.).

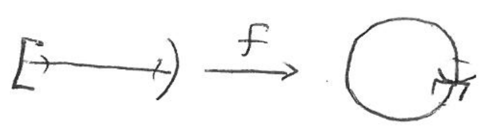
Ex | $f: \underbrace{[0,1)}_{\text{with s.s. top}} \to \underbrace{\{(x,y) \mid x^2+y^2=1\}}_{\text{with s.s. top.}}$

$$f(t) = (\cos(2\pi t), \sin(2\pi t))$$

f is continuous

f is bijective

f^{-1} is <u>not</u> continuous



Topology Day 8

Outline
- Continuity
- Topologies on products.

Announcements
- By popular demand, Mid. 1 on Feb. 27
- Exam will cover everything up to and including todays lec.

Motivating Question: What is the proper notion of equivalence for top. spaces?

Category	Equivalence
Groups	Isomorphism
Vector spaces	bijective linear map with linear inverse.
Top. Space	Homeomorphism

Def Let X and Y be top. spaces and $f: X \to Y$. f is a <u>homeomorphism</u> if f is a bijective continuous map with continuous inverse.

Note: X and Y are <u>homeomorphic</u> if there exists a homeomorphism $f: X \to Y$.

Ex $(\frac{-\pi}{2}, \frac{\pi}{2})$ is homeomorphic to $\mathbb{R}$.

$f: (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ $\quad f(x) = \tan(x)$

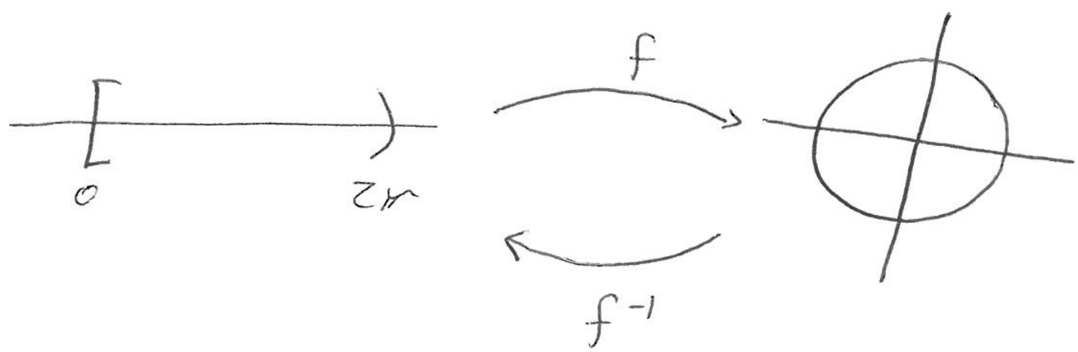
$\quad f^{-1}(x) = \tan^{-1}(x)$

From calculus we know each of these functions is continuous and bijective.

Note: f^{-1} is continuous is an important property of being a homeomorphism.

Ex| $f: [0, 2\pi) \to S^1$ (the unit circle in $\mathbb{R}^2$)

$f(x) = (\sin(x), \cos(x))$ is a continuous bijective map



However $f^{-1}([0,1))$ is not open in S^1, so f^{-1} is not continuous.

Basic constructions for continuous functions

Thm| Assume X, Y, Z are top. spaces

i) If $A \subset X$ is a subspace, then $i: A \to X$ s.t. $i(a) = a$ is continuous

ii) If $f: X \to Y$ and $g: Y \to Z$ are continuous, then $g \circ f: X \to Z$ is continuous.

iii) If $f: X \to Y$ is continuous and $A \subset X$ a subspace then $f|_A : A \to Y$ is continuous.

iv) If $f: X \to Y$ and $X = \bigcup_{\alpha \in J} U_\alpha$ for U_α open in X and $f|_{U_\alpha}$ is continuous for all α, then $f: X \to Y$ is continuous.

Pf | i) Let $f: A \to X$ by $f(a) = a$ for $A \subset X$ a subspace.

Let $U \subset X$ be open $f^{-1}(U) = U \cap A$.

By def. of subspace top. $f^{-1}(U)$ is open in A.

Hence, f is continuous. $\square$

ii) WTS $g \circ f: X \to Z$ is continuous.

Let $U \subset Z$ be an open set.

$g^{-1}(U)$ is open in Y by continuity of g.

$f^{-1}(g^{-1}(U))$ is open in X by cont. of f.

By set theory $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$.

Hence, $(g \circ f)^{-1}(U)$ is open in X and $g \circ f$ is cont. $\square$

iii) WTS $f|_A : A \to Y$ is continuous.

~~Let U be open in Y~~ Note: $f|_A = f \circ i : A \to Y$

where $i: A \to X$ is the inclusion map. ~~and~~

By i) and ii) $f|_A$ is continuous.

iv) Let $X = \bigcup_{\alpha \in J} U_\alpha$ and suppose $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for all $\alpha \in J$. Let $V \subset Y$ be an open set. $f|_{U_\alpha}^{-1}(V)$ is open for all $\alpha \in J$.

So $\bigcup_{\alpha \in J} f|_{U_\alpha}^{-1}(V)$ is open

However, $f|_{U_\alpha}^{-1}(V) = \{x \in U_\alpha \mid f|_{U_\alpha}(x) \in V\}$

$$= \{x \in X \mid f(x) \in V \text{ and } x \in U_\alpha\}$$

$$= f^{-1}(V) \cap U_\alpha .$$

So $\bigcup_{\alpha \in J} f|_{U_\alpha}^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(V) \cap U_\alpha$

(by dist. law) $= f^{-1}(V) \cap \left(\bigcup_{\alpha \in J} U_\alpha\right)$

$$= f^{-1}(V) \cap X$$

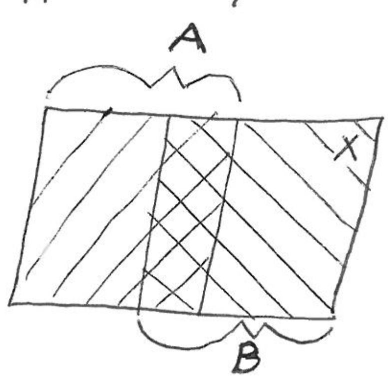
$$= f^{-1}(V)$$

Hence, f is continuous. $\square$

* Important *

Pasting Lemma: Let X be a top. space with $A, B \subset X$ closed subsets. s.t. $X = A \cup B$. Let $f: A \to Y$ and $g: B \to Y$ be continuous s.t. $f(x) = g(x)$ for all $x \in A \cap B$. Then the function

$$h: X \to Y \qquad h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases} \text{ is continuous.}$$



Pf] We will show h^{-1} of a closed set is closed.

Let $C \subset Y$ be closed.

$$h^{-1}(C) = \{x \in X \mid h(x) \in C\}$$
$$= \{x \in A \mid f(x) \in C\} \cup \{x \in B \mid g(x) \in C\}$$
$$= f^{-1}(C) \cup g^{-1}(C)$$

By continuity of f and g, $f^{-1}(C)$ is closed in A and $g^{-1}(C)$ is closed in B.

By Munkres 17.2, $f^{-1}(C) = A \cap K_1$ and $g^{-1}(C) = B \cap K_2$ where K_1 and K_2 are closed in X. Since finite unions and arbitrary intersections of closed are closed, then $h^{-1}(C)$ is closed. Hence h is continuous. $\square$

Topologies on Products

Recall: Given top. spaces X and Y a product topology on $X \times Y$ has basis $\mathcal{B} = \{U \times V \mid U \text{ open in } X \text{ and } V \text{ open in } Y\}$

What about infinite products?

Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of top. spaces.

$$\prod_{\alpha \in J} X_\alpha = \left\{ \vec{x} : J \longrightarrow \bigcup_{\alpha \in J} X_\alpha \;\middle|\; \vec{x}(\alpha) \in X_\alpha \text{ for all } \alpha \in J \right\}$$

Ex| $J = \mathbb{Z}^+$, so $\prod_{\alpha \in J} X_\alpha = \prod_{i=1}^{\infty} X_i = X_1 \times X_2 \times \cdots$

$\vec{x} \in \prod_{i=1}^{\infty} X_i$ is a function $\vec{x} : \mathbb{Z}^+ \longrightarrow \bigcup_{i=1}^{\infty} X_i$.

($\prod_{i=1}^{\infty} \mathbb{R}$ is the set of all sequences in $\mathbb{R}$).

Def| Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of top. spaces
The box topology on $\prod_{\alpha \in J} X_\alpha$ is given
by the basis $\mathcal{B}_{box} = \left\{ \prod_{\alpha \in J} U_\alpha \;\middle|\; U_\alpha \text{ is an open subset of } X_\alpha \right\}$

Exercise: Check that $\mathcal{B}_{box}$ is a basis

Def| The product topology on $\prod_{\alpha \in J} X_\alpha$ is given by
the basis $\mathcal{B}_{prod.} = \left\{ \prod_{\alpha \in J} U_\alpha : U_\alpha \subset X_\alpha \text{ is open } \underline{\text{and}}\right.$
$\left. U_\alpha = X_\alpha \text{ for all but finitly many } \alpha \right\}$.

Exercise| Check that $\mathcal{B}_{prod}$ is a basis

Note: the box topology is finer than the product topology.