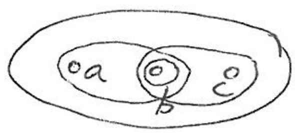
Topology Day 6

Outline

- Hausdorff Spaces
- Separation axioms (T_0, T_1, T_2, T_3, T_4)
- Continuity.

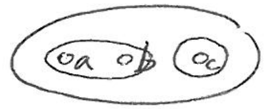
Motivating Questions

Q: In a top. space X, is it always true that a one pt set $\{x\}$ is closed?

No:  $\{b\}$ is not closed.

Q: In a top. space X, is it always true that a convergent sequence has a unique limit?

($\{X_n\}$ ~~is con~~ converges to x if for every nbh U of x $\exists N > 0$ s.t. $X_n \in U$ for $n \geq N$.)

No, In  the sequence a, b, a, b, a, b converges to both a and b.

We can avoid these problems by restricting to Hausdorff spaces.

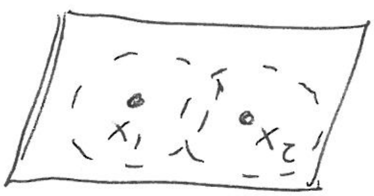
<u>Def</u> A top. space X is Hausdorff if for every $x_1, x_2 \in X$ s.t. $x_1 \neq x_2$ there exist <u>disjiont</u> open sets U_1 and U_2 s.t. $x_1 \in U_1$ and $x_2 \in U_2$.

The separation axioms are a list of increasingly restrictive properties that a "nice" top. space can have

$$\begin{array}{c} T_4 \\ \Downarrow \\ T_3 \\ \Downarrow \\ T_2 \text{ (Hausdorff).} \\ \Downarrow \\ T_1 \\ \Downarrow \\ T_0 \end{array}$$

<u>Def</u> A top. space X is <u>T_1</u> if given $x_1, x_2 \in X$ s.t. $x_1 \neq x_2$, there exist open nbhs U_1 of x_1 and U_2 of x_2 s.t. $x_2 \notin U_1$ and $x_1 \notin U_2$.

i.e.



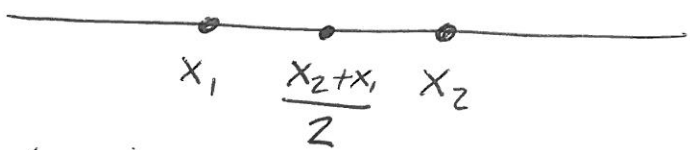
So, T_2 obviously implies T_1.

<u>Prop</u> (HW) A top. space X is T_1 iff all one point sets are closed.

<u>Ex</u> Let $\mathcal{T} = \{ U \subset \mathbb{R} \mid \mathbb{R} - U \text{ is finite} \}$.
$(\mathbb{R}, \mathcal{T})$ is a top. space that is T_1, but not T_2.

Ex Show that $\mathbb{R}$ is Hausdorff.

Let $x_1, x_2 \in \mathbb{R}$ s.t. $x_1 \neq x_2$.
With out loss of generality assume $x_1 < x_2$.



~~(x_1, x_2+1) is an open set containing x_2~~
~~(x_1+1, x_2) is an open set containing x_1~~

$\left(\frac{x_1+x_2}{2}, x_2+1\right)$ is an open set containing x_2.
$\left(x_1-1, \frac{x_1+x_2}{2}\right)$ is an open set containing x_1.

$$\left(\frac{x_1+x_2}{2}, x_2+1\right) \cap \left(x_1-1, \frac{x_1+x_2}{2}\right) = \emptyset.$$

Hence we have constructed disjiont open sets containing x_1 and x_2 respectivily. $\square$

Prop In a Hausdorff space X,
1) one-point sets are closed
2) limits of sequences are unique.

Pf Let $x \in X$. We will show $\overline{\{x\}} = \{x\}$.
Let $y \in X$ s.t. $x \neq y$. Since X is Hausdorff $\exists\, U_x, U_y \subset X$ open sets s.t. $x \in U_x$ and $y \in U_y$ and

$\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$. Recall that $x \in \overline{A}$ iff for every nbh $\mathcal{U}$ of x $\mathcal{U} \cap A \neq \emptyset$. Since $\mathcal{U}_y$ is a nbh of y s.t. $\mathcal{U}_y \cap \{x\} = \emptyset$, then $y \notin \overline{\{x\}}$. Hence, $\overline{\{x\}} = \{x\}$. $\square$

2) Suppose $\{x_n\}$ converges to x.
We will show that if $y \neq x$ then $\{x_n\}$ does not converge to y.
Since $y \neq x$, $\exists\, \mathcal{U}_x$ and $\mathcal{U}_y$ open in X s.t. $y \in \mathcal{U}_y$, $x \in \mathcal{U}_x$ and $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$.
By def. of convergent, $\exists\, N > 0$ s.t. if $n \geq N$, then $x_n \in \mathcal{U}_x$. Hence $\mathcal{U}_y$ is a nbh of y that contains only finitly many elements in the sequence $\{x_n\}$, thus $\{x_n\}$ does not converge to y. $\square$

Exercises | ① A subspace of a Hausdorff space is Hausdorff.
② A product of a Hausdorff space is Hausdorff.

$(\mathbb{R}, \mathcal{T})$ is T_1. If $x \neq y$ then $\mathbb{R}-\{x\}$ and $\mathbb{R}-\{y\}$ meet def. of T_1.

$(\mathbb{R}, \mathcal{T})$ is <u>not</u> T_2. All non empty open sets in $\mathcal{T}$ intersect.

We will return to do more on separation axioms in a few weeks

Continuity

Motivating Question: How do we map between top. spaces?

Category	Morphism ("natural" maps)
Vector spaces	Linear Maps
Groups	Homomorphisms
Top. Spaces	Continuous functions.

Def A function $f: X \to Y$ between top. spaces X and Y is continuous if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.

(Recall $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$.

Ex The identity map is ~~open~~ continuous.

$$f: (X, \mathcal{T}) \to (X, \mathcal{T}) \text{ s.t. } f(x) = x.$$

Let U be an open set in X (thought of as the codomain).

$f^{-1}(U) = U$ ~~s.t.~~ since $f(x) = x$.

Hence, $f^{-1}(U)$ is open in X (thought of as the domain)

Ex| Let τ be the standard topology on $\mathbb{R}$.
Let δ be the discrete topology on $\mathbb{R}$.

Claim: $f: (\mathbb{R}, \tau) \to (\mathbb{R}, \delta)$ is <u>not</u> continuous.
$f(x) = x$.

Claim: $f: (\mathbb{R}, \delta) \to (\mathbb{R}, \tau)$ <u>is</u> continuous
$f(x) = x$.