Topology Day 5

Outline
- Closed sets
- Closure
- Limit points

Recall: $Y \subset X$ is closed iff $X - Y$ is open.

Key facts about closed sets

Prop: Let X be a top. space
0) X and $\emptyset$ are closed sets
1) Arbitrary intersections of closed sets are closed.
2) Finite ~~intersections~~ unions of closed sets are closed

Pf) 0) $\emptyset$ is open and $X = X - \emptyset$.
X is open and $\emptyset = X - X$.

1) Suppose $\{A_\alpha\}_{\alpha \in J}$ is a collection of closed sets.
Examine $X - (\bigcap_{\alpha \in J} A_\alpha) = \bigcup_{\alpha \in J} (X - A_\alpha)$, by de Morgan's Law.
Since A_α is closed, $X - A_\alpha$ is open for each $\alpha \in J$.
Since $X - A_\alpha$ is open for each $\alpha \in J$, then $\bigcup_{\alpha \in J} (X - A_\alpha)$ is open.
Since $\bigcup_{\alpha \in J} (X - A_\alpha)$ is open, then $\bigcap_{\alpha \in J} A_\alpha$ is closed. $\square$

2) Similar to 1).

Q: How do closed subsets relate to the subspace topology?

Prop Let $A \subset Y \subset X$ where X is a top. space. A is a closed set in Y with the subspace top. iff $A = Y \cap C$ where C is closed in X.

Pf $\Rightarrow$) Suppose A is closed in Y.

Thus, $Y - A$ is open in Y.

By def. of subspace top., $Y - A = Y \cap U$ where U is open in X.

Hence, $X - U$ is closed.

Claim: $A = Y \cap (X - U)$

Exercise

$\Leftarrow$) Suppose C is closed in X and $A = Y \cap C$.

Thus, $X - C$ is open in X.

By def. of subspace top $Y \cap (X - C)$ is open in Y. Hence, $Y - (Y \cap (X - C))$ is closed in Y.

Claim $A = Y - (Y \cap (X - C))$

Exercise

$\square$

Closure

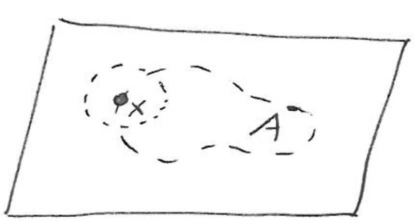
Def Given $A \subset X$, the <u>closure</u> of A, denoted $\bar{A}$, is the intersection of all closed sets that contain A.

Facts
- $\bar{A}$ is closed
- $A \subset \bar{A}$
- $A = \bar{A}$ iff A is closed (exercise)
- $\bar{A}$ is the "smallest" closed set containing A.
- $\bar{X} = X$, $\bar{\emptyset} = \emptyset$.

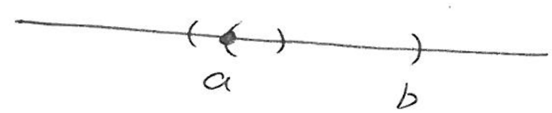
The following is a usefull alternative characterization.

Prop Let $A \subset X$. $x \in \bar{A}$ iff every open set U that contains x intersects A non-trivially.

Pic



Ex In $\mathbb{R}$,
$a \in \overline{(a, b)}$



Pf (Instead of "$P \Leftrightarrow Q$", we will show "not $P \Leftrightarrow$ not Q".)

Show $x \notin \bar{A}$ iff there exists $U \subset X$ open s.t. $U \cap A = \emptyset$.

$\Rightarrow$| If $x \notin \bar{A}$, then $x \in X - \bar{A}$. Since $\bar{A}$ is closed, $X - \bar{A}$ is open. Thus $X - \bar{A}$ is an open set that contains x and is disjiont from A.

$\Leftarrow$| Suppose there exists an open set $\mathcal{U} \subset X$ s.t. $x \in \mathcal{U}$ and $\mathcal{U} \cap A = \emptyset$. Since $\mathcal{U}$ ~~set~~ is open $X - \mathcal{U}$ is closed. Since $\bar{A}$ is the intersection of all closed sets contaning A, and $x \notin X - \mathcal{U}$, then $x \notin \bar{A}$. $\square$

Useful Prop| If $(X, \mathcal{T})$ has basis $\mathcal{B}$, then $x \in \bar{A}$ iff every basis element containing x intersects A.

Examples| Let $\mathbb{R}_s$ be $\mathbb{R}$ with the standard topology.

In $\mathbb{R}_s$, $A = \{ 1/n \mid n \in \mathbb{Z}^+ \}$ $\bar{A} = A \cup \{0\}$

In $\mathbb{R}_s$, $\bar{\mathbb{Q}} = \mathbb{R}$.

In $\mathbb{R}_s$, $A = (a, b)$, $\bar{A} = [a, b]$

Note: There is something speacial about points in $\bar{A} - A$.

Def| Let X be a top. space and $A \subset X$. A point $x \in X$ is a limit point of A if every open set $\mathcal{U}$ that contains x intersects $A - \{x\}$ nontrivially. (Equivalently, x is a limit point of A if $x \in \overline{A - \{x\}}$).

Examples | • In $\mathbb{R}_s$, the limit pts of $(0,1]$ are all points in $[0,1]$.

• In $\mathbb{R}_s$, the limit points of $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ is just 0.

• In $\mathbb{R}_s$, the limit points of $\{0\}$ is $\emptyset$.

Let A' denote the set of limit points of a set A.

Thm | $\overline{A} = A \cup A'$.

Pf | ~~Exercise.~~

$\subseteq$ | Let $x \in \overline{A}$

Case 1: If $x \in A$ then $x \in A \cup A'$

Case 2: If $x \notin A$, then by prop. every open set containg x intersects A nontrivially. Since $x \in A$, then every open set containg x intersects $A - \{x\}$ nontrivially. So $x \in A'$

In either case $x \in A \cup A'$.

$\supseteq$ | First, $A \subset \overline{A}$ by def. of closure

Claim: $A' \subset \overline{A}$

Let $x \in A'$, then every open set that contains x intersects $A - \{x\}$ nontrivially.

So, every open set that contains x ~~and~~ intersects A non trivialy, by previous prop. $x \in \overline{A}$.

Hence $A \cup A' \subset \overline{A}$.