Topology 26

O.H. Th: 5:15 - 7:15
Fr: 10 - 11
Mon: 2 - 5 pm

- Path Connectedness

Annoucements

- Final exam a week from today
- Next time Advitizement for 550 B.

Def Given $x, y \in X$ a path in X from x to y is a continuous map $f: [0,1] \to X$ with $f(0) = x$ and $f(1) = y$. X is path connected if any two points con be jioned by a path.

Lemma If X is path connected, then X is connected.

Pf Suppose X is path connected and has a separation $X = A|B$. Pick $a \in A$ and pick $b \in B$. Since X is path connected there exists a continous function $f: [0,1] \to X$ with $f(0) = a$ and $f(1) = b$. Recall $[0,1]$ is connected and the continous image of connected is connected. Hence, $f([0,1])$ is connected. By a previous lemma 23.2, $f([0,1]) \subset A$ or $f([0,1]) \subset B$. Either conclusion contradicts the fact that $A \cap B = \emptyset$, $a \in A$ and $b \in B$. $\square$

Prop: If A is a connected subspace of X and $A \subset B \subset \overline{A}$, then B is connected.

Pf] Suppose B has a separation $B = C/D$. Since A is connected, then by Munkres 23.2, $A \subset C$ or $A \subset D$. WLOG say $A \subset C$. Hence, $\overline{A} \subset \overline{C}$ and $B \subset \overline{C}$. Thus $D \cap \overline{C} \neq \emptyset$. This contradicts Munkres 23.1 which says $D \cap \overline{C} = \emptyset$. So, B has no separation. $\square$

Key example

"Topologists sine curve"

$F: (0,1] \to \mathbb{R}^2$, $F(t) = (t, \sin(\frac{1}{t}))$

Claim] $F((0,1])$ is connected.

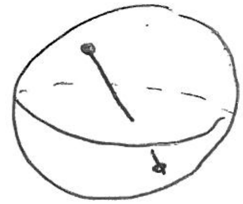
Pf] This follows from the fact that $(0,1]$ is p.c. and F is continuous.

Claim] $\overline{F((0,1])} = F((0,1]) \cup (\{0\} \times [-1,1])$

Pf] exercise.

Examples of path connected spaces

- $\mathbb{R}^n$
- $B^n = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq 1 \}$



- Any "convex" subset of $\mathbb{R}^n$
- $\mathbb{R}^n - \{\vec{0}\}$ for $n \geq 2$.

<u>Lemma</u> The continuous image of a path connected space is path-connected.

<u>Pf</u> Let X be path connected and let $f: X \to Y$ be a continuous map. Let $a, b \in f(X)$.

Pick $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$.
Since X is path connected there is a path $g: [0,1] \to X$ from x to y. Since f and g are continous, then

$$f \circ g : [0,1] \to Y \text{ is a path from } a \text{ to } b. \quad \square$$

<u>Claim</u> S^n for $n \geq 1$ is path connected.

$$f: \mathbb{R}^n - \{\vec{0}\} \to S^{n-1}, \quad f(\vec{x}) = \frac{\vec{x}}{\|\vec{x}\|}$$

is a continuous map. Since $\mathbb{R}^n - \{\vec{0}\}$ is p.c. for $n \geq 2$ then S^k is p.c. for $k \geq 1$. $\square$

Note: by our previous prop, $\overline{F((0,1])}$ is connected.

Claim $\overline{F((0,1])}$ is not path connected.

Pf Suppose, to form a contradiction, that there is a path g from some point $a \in \{0\} \times [-1,1]$ to some point b in $F((0,1])$. We can choose to represent g as a paramitrization $g(t) = \langle x(t), y(t) \rangle$

WLOG we can assume $x(t) > 0$ for $t > 0$.

Hence, $y(t) = \sin\left(\frac{1}{x(t)}\right)$ for $t > 0$.

Note $\{\frac{1}{n}\} \to 0$ as $n \to \infty$

For each $n \geq 1$ there exist $u_n \in (0, x(\frac{1}{n}))$

$$\sin\left(\frac{1}{u_n}\right) = (-1)^n$$

Now x is continuous, so $\exists\, t_n \in (0, \frac{1}{n})$ s.t. $x(t_n) = u_n$

Thus $y(t_n) = \sin\left(\frac{1}{x(t_n)}\right) = (-1)^n$

Since $t_n \to 0$ but $y(t_n)$ does not converge we have a contradiction to continuity of y. $\square$

Topology 27

- Preview of 5501B

Announcements
- Final Tues. May 13th 5-7pm
- 8 questions

Recall: A path in X is a continuous map $f:[0,1]\to X$

<u>Def</u> Given $x_0 \in X$, a <u>loop in X based at x_0</u> is a path $f:[0,1]\to X$ s.t. $f(0)=f(1)=x_0$.

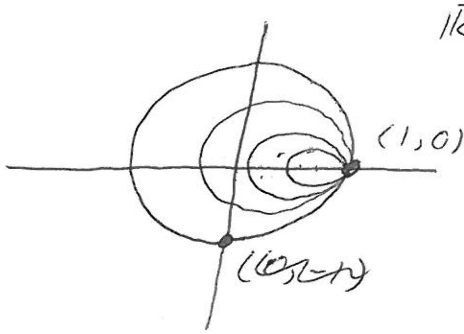
We want a natural notion of equivalence of based loops.

<u>Def</u> Two loops f,g based at $x_0\in X$ are <u>homotopic</u> if there exists a continuous function

$$\boxed{F:[0,1]\times[0,1]\to X \text{ s.t. } F(0,t)=f(t) \text{ and } F(1,t)=g(t). \quad F(s,0)=x_0=F(s,1).}$$

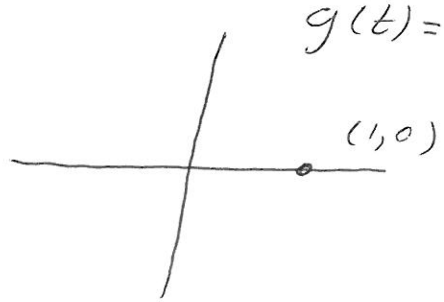
<u>Ex</u> $f(t)=\langle \cancel{\sin(2\pi t)}\ \cos(2\pi t),\ \cancel{\cos(2\pi(t))}\ \sin(2\pi t)\rangle$

$\mathbb{R}^2$



path homotopic to <u>constant</u> based loop at $(0,-1)$

$g(t)=\langle 1,0\rangle$



$$F(s,t)=\langle (1-s)\cos(2\pi t)+s,\ (1-s)\sin(2\pi t)\rangle$$

There is a binary operation on based loops called contatenation (or stacking).

If f and g are loops based at x_0, then

$$f * g : [0,1] \to X$$

$$f * g(t) = \begin{cases} g(2t) & 0 \le t \le \frac{1}{2} \\ f(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Let Y be the set of loops based at x_0.

Let $\sim$ be the equivalence relation on Y that $f \sim g$ if f is homotopic to g.

<u>Claim</u> | $Y/\sim$ under the binary operation of stacking is a group!

Recall group axioms

1. Closure ($f*g$ is still a loop based at x_0)
2. Identity element (the constant loop is the identity)
3. Inverse element ($f(1-t)$ is the inverse of $f(t)$)
4. accociativity ($f*(g*h) = (f*g)*h$).

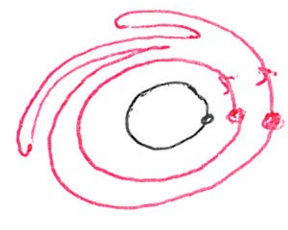
$Y/\sim$ is denoted $\pi_1(X, x_0)$ and is known as the fundamental group of X based at x_0.

<u>Examples</u>

$$\pi_1(\mathbb{R}^2, (0,0)) \cong \{1\}$$

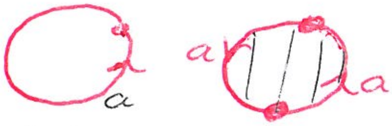
$$\pi_1(S^1, (1,0)) \cong \mathbb{Z}$$

take a week



$$\pi_1(S^1 \times S^1, (1,0)\times(1,0)) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$\pi_1($  $) \cong \pi_1(D^2) \cong \{1\} \cong \dfrac{\langle a \rangle}{\langle\langle a = 1 \rangle\rangle}$

$\pi_1($  $) \cong \pi_1(?) \cong \dfrac{\langle a \rangle}{\langle\langle a^2 = 1 \rangle\rangle} = \langle a \mid a^2 = 1 \rangle$

$\mathbb{R}P^2$

This is called a group presentation

$$\langle \underbrace{a}_{\text{generators}} \mid \underbrace{a^2 = 1}_{\text{relations}} \rangle \cong \mathbb{Z}_2$$

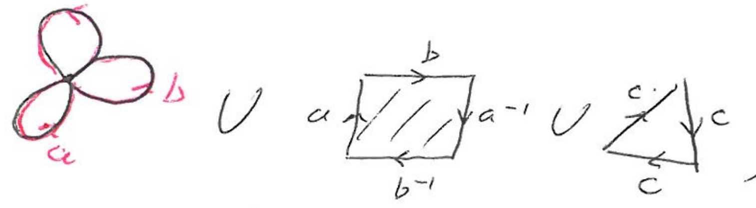
$\pi_1($  $) \cong \pi_1(S^1 \vee S^1) \cong \langle a, b \mid \ \rangle \cong F_2$

the free group on two variables.

So, $\pi_1(\bigvee_{i=1}^{n} S^1) \cong F_n$ (free group on n-genators)

<u>Def)</u> A group is finitly presented if it can be constructed from a finite list of genators and a finite list of relations.

$$\langle a, b, c \mid aba^{-1}b^{-1} = 1,\ c^3 = 1 \rangle$$

$\pi_1($  $) \cong$

$$\langle a, b, c \mid aba^{-1}b^{-1} = 1,\ c^3 = 1 \rangle$$

Topology contains <u>all</u> group theory

2-dimentional CW complexes can all be embedded in $\mathbb{R}^5$. So, subspaces of $\mathbb{R}^5$ contain all group theory.