Topology Day 19

Outline

- Finishing Quotient Spaces
- Countability Axioms

Announcements

- Exam 5Q, 1Q on definitions
- Exam on Tuesday April 15th.

<u>Recall</u> Lemma: If X is a top space, ~~and~~ Y is a set and $p: X \to Y$ is surjective, then $\exists!$ topology on Y s.t. $p: X \to Y$ is a quotient map.

(Define $V \subset Y$ to be open if $p^{-1}(V)$ is open)

Given an equivalence relation $\sim$ on X there is a natural quotient topology to define on $X/\sim$ (the set of equivalence classes of $\sim$ on X).

Be careful with Quotient maps and topologies

Fact: If $p: X \to Y$ is a quotient map and $A \subset X$ is a subspace, $p|_A$ need not be a quotient map.

<u>Ex</u> We have seen that $\pi_1 : X \times Y \to X$ s.t. $\pi_1(x,y) = x$ is a quotient map.

Let $A = \{(0,0)\} \cup \{(x, 1/x) \mid x \neq 0\} \subset \mathbb{R}^2$

Examine $\pi_1|_A$.

Note $\{(0,0)\} \subset A$ is open in A

But, $\{0\} \subset \mathbb{R}$ is not open

Since $(\pi_1|_A)^{-1}(0) = \{(0,0)\}$ this contradicts p is a quotient map.

Fact | If $p_1 : X_1 \to Y_1$ and $p_2 : X_2 \to Y_2$ are quotient maps, then $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ need not be a quotient map. (pg. 143 Example 7)

Lemma | If $p : X \to Y$ and $q : Y \to Z$ be quotient maps, then $q \circ p : X \to Z$ is a quotient map.

Pf | Suppose $V \subset Z$ is open wts $(q \circ p)^{-1}(V)$ is open.

$$(q \circ p)^{-1}(V) = p^{-1}(q^{-1}(V))$$

Since q is a quotient map $q^{-1}(V)$ is open in Y.

Since p is " " $p^{-1}(q^{-1}(V))$ is open in X.

Hence, $(q \circ p)^{-1}(V)$ is open.

Suppose $(q \circ p)^{-1}(V)$ is open.

$p^{-1}(q^{-1}(V))$ is open.

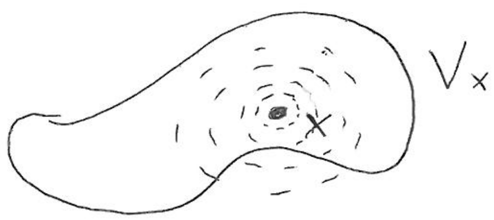
Since p is a quotient map $q^{-1}(V)$ is open.

Since q is a quotient map V is open.

Thus $q \circ p$ is a quotient map.

Countability Axioms

Motivation: If (X,d) is a metric space with the metric topology and $x \in X$ then $\exists$ a countable collection of nbhs of x $\{U_i\}_{i=1}^{\infty}$ s.t. Any nbh of x contains one of the U_i.



Def | A top. space X is first-countable, if given any $x \in X$, $\exists$ a countable collection of nbhs of x given by $\{U_i\}_{i=1}^{\infty}$ s.t. any nbh of x contains some U_i.

Prop | If X is first-contable

a) Given $A \subset X$, $x \in \bar{A}$ iff $\exists$ a seq. $\{x_n\} \subset A$ s.t. $x_n \to x$.

b) Given $f: X \to Y$, f is cont. iff $\forall x_n \to x$ in X, we have $f(x_n) = f(x)$.

Proof | Same as for metric spaces

Def | A top. space X is 2nd-countable if $\exists$ a basis $\mathcal{B}$ for X s.t. $\mathcal{B}$ is countable

Note: Second Countable $\Rightarrow$ first-countable
(Let U_n be the collection of all basis elements that contain x.)

Ex| $\mathbb{R}$ with the standard topology is 2nd-countable.

Let $\mathcal{B} = \{ (a,b) \mid a<b \text{ and } a,b \in \mathbb{Q} \}$

$\mathcal{B}$ is a basis for $\mathbb{R}$. (H.W. 1)

Why is $\mathcal{B}$ countable?

$P: \mathcal{B} \to \mathbb{Q} \times \mathbb{Q}$ s.t. P is 1-1.

$P((a,b)) = (a,b)$

if $(a_1, b_1) = (a_2, b_2)$ as points in $\mathbb{Q}\times\mathbb{Q}$

then $(a_1, b_1) = (a_2, b_2)$ as open intervals.

Hence, $\mathcal{B}$ is in bijection with a subset of a countable set,

So, $\mathcal{B}$ is countable.

Ex| $\mathbb{R}_\ell$ is not second-countable. Let $\mathcal{B}$ be any basis.

Recall $[x, x+1)$ is open in $\mathbb{R}_\ell$ for all x.

Hence $\exists\, B_x \in \mathcal{B}$ s.t. $x \in B_x \subset [x, x+1)$

However, if $x \neq y$ then $B_x \neq B_y$.

(Since $\inf(B_x) = x \neq y = \inf(B_y)$)

So, $\mathcal{B}$ has uncountably many elements.

Ex| A metric space need not be 2nd-countable.

Let $\mathbb{R}$ have the discrete metric.

Since any basis for $\mathbb{R}_d$ must contain the singletons.