Thm (Heine-Borel)

A subspace $K \subset \mathbb{R}^n$ is compact iff K is closed and bounded

Pf $\Rightarrow$ Suppose $K \subset \mathbb{R}^n$ is compact. Since $\mathbb{R}^n$ is Hausdorff, then K is closed.

Suppose to form a contradiction that K is unbounded. Then $\{U_k\}_{k \in \mathbb{Z}^+}$ s.t.

$U_k = \underbrace{(-k,k) \times (-k,k) \times \cdots \times (-k,k)}_{n \text{ times}}$ is an open cover of K with no finite sub cover. Hence K must be bounded.

$\Leftarrow$ Suppose K is closed and bounded.

Since K is bounded there exists a closed box

$B = \underbrace{[-L,L] \times \cdots \times [-L,L]}_{n\text{-times}}$ s.t. $K \subset B$.

Since the product of compact spaces is compact, then B is compact.

Since closed subsets of compact spaces are compact then K is compact. (Here we are implicity using the H.W. problem that states subspace top. on $K \subset \mathbb{R}^n$ is same as $K \subset B \subset \mathbb{R}^n$.)

Thm| (Extreem value thm)
Let $f: X \to \mathbb{R}$ be continuous, with X compact. Then f takes on its maximum and minimum values.

Pf| Since the continuous image of compact is compact, then $f(X)$ is a compact subset of $\mathbb{R}$. By Heine Borel, $f(X)$ is closed and bounded. Let $M = \sup(f(X))$. Since $f(X)$ is bounded, $M < \infty$. By def. of supremum $\exists \{a_n\}_{n=1}^{\infty}$ contained in $f(X)$ s.t. $\lim_{n\to\infty} a_n = M$.
Since $f(X)$ is closed $\overline{f(X)} = f(X)$ and $M \in f(X)$. Thus f atains its maximum.
A similar argument shows f attains its minimum. $\square$

Other notions of compactness

Def| X is limit point compact if every infinite subset of X has a limit point.

Ex| $\mathbb{R}$ is not l.p. compact since $\mathbb{Z} \subset \mathbb{R}$ has no limit points.

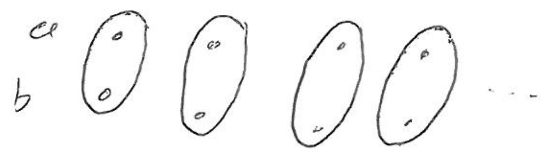
Prop| If X is compact then X is limit point compact.

Pf| Exercise

Ex| $X = (\mathbb{Z}_+, \text{discrete}) \times (\{a,b\}, \text{indiscrete})$

Claim| X is not compact.

Look at the open cover $U_n = \{n\} \times Y$



Claim| X is l.p. compact

Pf| Let A be any non empty set.
Since A is non-empty $\exists n \in \mathbb{Z}^+$ s.t. (n,a) or $(n,b) \in A$. If $(n,a) \in A$, then (n,b) is a limit point of A. If $(n,b) \in A$, then (n,a) is a limit point of A.

Def| X is sequentially compact if every sequence in X has a convergent subsequence.

In general, not comparable to compactness.

Thm| If X is metrizable, TFAE

1) X is compact
2) X is limit point compact
3) X is sequentially compact

Pf| See any real analysis text.

Def Let X be a top. space. A collection $\mathcal{C}$ of subsets of X has the <u>finite</u> <u>intersection</u> <u>property</u> if if for every finite sub collection $\{C_1, \dots, C_n\} \subset \mathcal{C}$ the intersection $\bigcap_{i=1}^{n} C_i \neq \emptyset$.

Thm X is compact iff every collection of closed sets $\mathcal{C}$ with the finite intersection property also has the property $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

~~Pf~~

Ex $X = (0,1]$ and $\mathcal{C} = \{(0,\tfrac{1}{n}] \mid n \in \mathbb{Z}^+\}$

The sets in $\mathcal{C}$ are closed in X and have the finite intersection property. However $\bigcap_{n \in \mathbb{Z}^+} (0,\tfrac{1}{n}] = \emptyset$

Pf First some observations

Let $\mathcal{U}$ be any collection of sets in X and

$\mathcal{C} = \{X - U \mid U \in \mathcal{U}\}$.

1. elements of $\mathcal{U}$ are open $\Leftrightarrow$ elements of $\mathcal{C}$ are closed
2. $\mathcal{U}$ covers X $\Leftrightarrow$ $\bigcap_{C \in \mathcal{C}} C = \emptyset$
3. $\{U_1, \dots, U_n\} \subset \mathcal{U}$ covers $X \Leftrightarrow C_1 \cap \dots \cap C_n = \emptyset$.

(contrapositive of the definition of compactness)

X is compact iff "for any collection $\mathcal{U}$ of open sets, if no finite subcover of $\mathcal{U}$ covers X then $\mathcal{U}$ does not cover X."

~~tt~~ This statement is equivalent to.

"for any collection $\mathcal{C}$ of closed sets, if every finite intersection is non empty, then $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$"

$\square$

<u>Cor</u> If X is compact and $C_1 \supset C_2 \supset \cdots$ is a sequence of nested non empty closed sets, then $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

<u>Pf</u> Let $\mathcal{C} = \{C_n\}$. $\mathcal{C}$ has the finite intersection property, so $\bigcap_{n=1}^{\infty} C_n = \emptyset$.