Topology Day 14

Outline

- Compactness

Motivating Question From calculous we know every function $f:[a,b]\to\mathbb{R}$ attains its max. What topological property about $[a,b]$ makes this true?

Def Given a top. space X, an <u>open cover</u> of X is a collection of open subsets $\{U_\alpha\}_{\alpha\in J}$ s.t. $X=\bigcup_{\alpha\in J} U_\alpha$.

Def X is compact if for every open cover $\{U_\alpha\}_{\alpha\in J}$ of X there exists a finite subcover, i.e. $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ s.t. $X\subset \bigcup_{i=1}^{n} U_{\alpha_i}$.

<u>Compactness is a topological property</u>

Examples

— Is $\mathbb{R}$ compact?

No, look at $\{(-n,n)\}_{n\in\mathbb{Z}^+}$

— Is $(-1,1)$ compact?

No, it is homeomorphic to $\mathbb{R}$.

— Is $X=\{0\}\cup\{\frac{1}{n} \mid n\in\mathbb{Z}^+\}$ ~~compact~~ with the sub space topology in $\mathbb{R}_s$ compact?

Yes! Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of X.

$0\in U_\beta$ for some β.

Since $\{\frac{1}{n}\}_{n\in\mathbb{Z}^+}$ convers to 0, U_β contains all but finitely many ~~of~~ elements of $\{\frac{1}{n}\}_{n\in\mathbb{Z}^+}$

Let $U_{\alpha_1},\dots,U_{\alpha_m}$ be the elements of the cover that contain these finitly many elements of $\{\frac{1}{n}\}_{n\in\mathbb{Z}^+}$.

Hence $X\subset U_\beta\cup\left(\bigcup_{i=1}^{m}U_{\alpha_i}\right)$. $\square$

Is $[a,b]\subset\mathbb{R}_s$ compact?

Yes, proof next time.

Lemma) Let X be a top. space. $Y \subset X$ with the subspace topology is compact iff for every collection of open sets in X $\{U_\alpha\}_{\alpha \in J}$ s.t. $Y \subset \bigcup_{\alpha \in J} U_\alpha$ there is a finite sub collection whose union contains Y.

Pf) Exercise Munkres 26.1.

Thm 1) A closed subspace Y of a compact space X is compact.

Pf) Let $\{U_\alpha\}_{\alpha \in J}$ be a collection of open sets in X s.t. $Y \subset \bigcup_{\alpha \in J} U_\alpha$. Then $\{U_\alpha\}_{\alpha \in J}$ together with $X - Y$ is an open cover of X. Since X is compact this open cover has a finite sub cover $X - Y$ (maybe), $U_{\alpha_1}, U_{\alpha_2}, \ldots U_{\alpha_n}$. Hence $Y \subset \bigcup_{i=1}^{n} U_{\alpha_i}$ and by the lemma Y is compact. $\square$

Thm 2 If X is Hausdorff and $Y \subset X$ is compact then Y is a closed subset of X.

Pf WTS $X-Y$ is open. Let $x_0 \in X-Y$

Since X is Hausdorff, $\forall y \in Y$ $\exists\, \mathcal{U}_y$ a nbh of y and V_y a nbh of x s.t.

$\mathcal{U}_y \cap V_y = \emptyset$.

Since $Y \subset \bigcup_{y \in Y} \mathcal{U}_y$, then, by the lemma, there is a finite sub collection $\mathcal{U}_1, \dots, \mathcal{U}_n$ s.t. $Y \subset \bigcup_{i=1}^{n} \mathcal{U}_i$.

Note $\bigcap_{i=1}^{\infty} V_i$ is open, since the finite intersection of open is open. Also, since $V_i \cap \mathcal{U}_i = \emptyset$ then $\bigcap_{i=1}^{n} V_i \cap \left(\bigcup_{i=1}^{n} \mathcal{U}_i \right) = \emptyset$.

Since $Y \subset \bigcup_{i=1}^{n} \mathcal{U}_i$, then $\bigcap_{i=1}^{n} V_i$ is an open nbh of x_0 that is disjiont from Y.

Since x_0 was arbitrary, then $X-Y$ is open and Y is closed. $\square$

¡Very Important theorem!

Thm 3 If $f: X \to Y$ is continuous and X is compact, then $f(X) \subset Y$ is compact.

Pf Let $\{U_\alpha\}_{\alpha \in J}$ be a collection of open sets in Y s.t. $f(X) \subset \bigcup_{\alpha \in J} U_\alpha$.

Since f is continuous, $\{f^{-1}(U_\alpha)\}_{\alpha \in J}$ is an open cover of X. (Recall $X \subset f^{-1}(f(X)) \subset f^{-1}(\bigcup U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha)$.

Since X is compact there exists a finite sub cover of X given by $f^{-1}(U_{\alpha_1}), f^{-1}(U_{\alpha_2}), \ldots, f^{-1}(U_{\alpha_n})$.

$$f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})\right) = \bigcup_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \subset \bigcup_{i=1}^{n} U_{\alpha_i}$$

Hence, by the lemma, $f(X)$ is compact. □

Homeomorphism trick

Prop Suppose $f: X \to Y$ is bijective and continuous with X compact and Y Hausdorff, then $f^{-1}: Y \to X$ is continuous and f is a homeo.

Pf It suffices to show that the image of a closed set in X is closed in Y. Let $A \subset X$ be closed. By theorem 1, A is compact. By thm 3, $f(K)$ is compact, By thm 2, $f(K)$ is closed. □