Topology Day 12

Outline

- Uniform metric
  Uniform convergence
- Connectedness

Uniform metric

See day 11 notes

Connectedness

Let X be a top space.

<u>Def</u> A <u>separation</u> of X is a pair of non-empty, disjiont, open sets U and V s.t. $X = U \cup V$. We write it as $X = U | V$. X is <u>connected</u> if it has no separation.

<u>Note</u> Connectedness is a topological property (i.e. preserved under homeomorphism)

Given a Separation $X = U|V$, $U = X - V$ and $V = X - U$, So the sets U and V are both closed <u>and</u> open. Hence, X is connected iff the only <u>clopen</u> sets are X and $\emptyset$.

Pf| $\Rightarrow$| If X is connected and $A \subset X$ is clopen, then $X-A$ is clopen. Since $X = A \cup X-A$, A, and $X-A$ are open and $A \cap (X-A) = \emptyset$, it must be that A or $X-A$ is $\emptyset$. Hence $A = \emptyset$ or $A = X$.

$\Leftarrow$| (contrapositive) If $X = U | V$, then U is a non-empty clopen set and V is a non empty clopen set. Hence $U \neq \emptyset$ and $U \neq X$.

Examples|

X with the indescrete top. is connected

X with discrete top. is not connected (if $|X| \geq 2$)

Lemma| Let X be a top. space and $Y \subset X$ be a subspace. Given $A, B \subset Y$ non-empty disjiont sub sets s.t. $Y = A \cup B$ $Y = A | B$ iff $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. (these closures are in X).

Pf| $\Rightarrow$| Suppose $Y = A|B$. Then both A and B are clopen in Y.

$$A \underset{\substack{\uparrow \\ \text{since } A \\ \text{is closed in } Y}}{=} \text{closure of } A \text{ in } Y \underset{\substack{\uparrow \\ \text{by 17.4}}}{=} \bar{A} \cap Y$$

So $A = \bar{A} \cap Y$

$$A \cap B = (\bar{A} \cap Y) \cap B$$

$$A \cap B = \bar{A} \cap B$$

(A and B are disjiont) $\emptyset = \bar{A} \cap B$

Swap roles of A and B to get $\emptyset = \bar{B} \cap A$.

$\Leftarrow$ Suppose $\bar{A} \cap B = \emptyset$ and $\bar{B} \cap A = \emptyset$.

The closure of A in $Y \underset{17.4}{=} \bar{A} \cap Y = \bar{A} \cap (\cancel{Y} A \cup B)$

$$= (\bar{A} \cap A) \cup (\bar{A} \cap B)$$

$$= A \cup \emptyset$$

$$= A$$

Hence A is closed in Y, so B is open.

By a symetric argument B is closed in Y, so A is open.

Thus $Y = A/B$. $\square$

Topology Day 13

Outline

- Connectedness

Last time:

Lemma| Let $Y \subset X$ be a subspace. Then non-empty disjiont subsets $A, B \subset Y$ with $A \cup B = Y$ form a separation of Y iff $\bar{A} \cap B = \emptyset$ and $\bar{B} \cap A = \emptyset$. (closures in X).

Recall| $A, B \subset X$ form a separation if

① A and B are open

② $A \cap B = \emptyset$

③ $A \cup B = X$

④ $A \neq \emptyset$ and $B \neq \emptyset$.

Ex| $X = \mathbb{R}$, $Y = [0,1) \cup (1,2]$

Let $A = [0,1)$, $B = (1,2]$

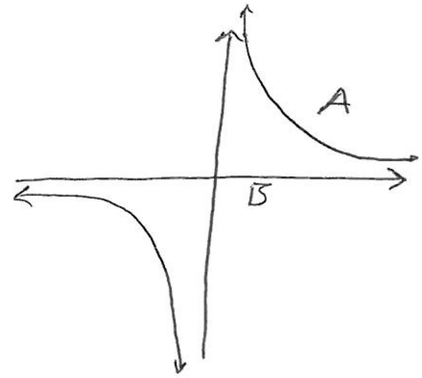
$A \cup B = Y$ $\quad$ $A \cap B = \emptyset$

$\bar{A} \cap B = \emptyset$ and $\bar{B} \cap A = \emptyset$.

So, By the lemma $Y = A | B$.

Ex $X = \mathbb{R}^2$ A = graph of $f(x) = 1/x$ on domain $\mathbb{R} - \{0\}$

B = x-axis



$Y = A \cup B$

$A \cap B = \emptyset$

$\bar{A} \cap B = \emptyset$

$\bar{B} \cap A = \emptyset$

By lemma, $Y = A | B$

Thm The image of a connected space under a continuous map is connected.

Pf Let X be a connected top. space and $f: X \to Y$ be a continuous map. Let $Z = f(X)$.

Hence, $f: X \to Z$ is continuous and onto.

Suppose to form a contradiction $Z = U | V$.

By continuity $f^{-1}(U)$ is open, $f^{-1}(V)$ is open

Since U and V are non empty and f is onto,

$f^{-1}(U)$ and $f^{-1}(V)$ are non empty.

Since, $Z = U \cup V$, $X = f^{-1}(U) \cup f^{-1}(V)$.

Examine $f^{-1}(U) \cap f^{-1}(V) \overset{\text{set theory fact}}{=} f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$.

Hence, $X = f^{-1}(U) | f^{-1}(V)$, a contradiction.

Hence, Z is connected.

Cor| If $f: X \to Y$ is onto and continuous and X is connected, then Y is connected.

Lemma| If $X = A|B$ and Y is a connected subspace of X then $Y \subset A$ or $Y \subset B$.

Pf| Suppose to form a contradiction that $Y \cap A \neq \emptyset$ and $Y \cap B \neq \emptyset$.

Since A and B are open in X, then $Y \cap A$ and $Y \cap B$ are open in Y.

Since $Y \subset X = A \cup B$, then $Y = (Y \cap A) \cup (Y \cap B)$.

Since $A \cap B = \emptyset$, then ~~Y~~ $\emptyset = (Y \cap A) \cap (Y \cap B)$.

Thus $Y = (Y \cap A) | (Y \cap B)$, a contradiction.

Hence $Y \cap A = \emptyset$ or $Y \cap B = \emptyset$. Since $X = A \cup B$, $Y \subset A$ or $Y \subset B$. $\square$

Prop 1| Let $\{A_\alpha\}_{\alpha \in J}$ be a family of connected subsets of X s.t. $a \in \bigcap_{\alpha \in J} A_\alpha$. Then $\bigcup_{\alpha \in J} A_\alpha$ is connected.

Pf| Suppose to form a contradiction that $\bigcup_{\alpha \in J} A_\alpha = C|D$.

Since $a \in \bigcup_{\alpha \in J} A_\alpha$, $a \in C$ or $a \in D$. WLOG say $a \in C$.

By the previous lemma, since A_α is connected and $A_\alpha \cap C \neq \emptyset$, then $A_\alpha \subset C$ for every α.

This is a contradiction to $D \neq \emptyset$. $\square$.

Prop 2 If X and Y are connected, then $X \times Y$ is connected.

Pf **Claim** for any $a \in X$ and $b \in Y$

$X \times \{b\}$ is homeomorphic to X

$\{a\} \times Y$ is homeomorphic to Y.

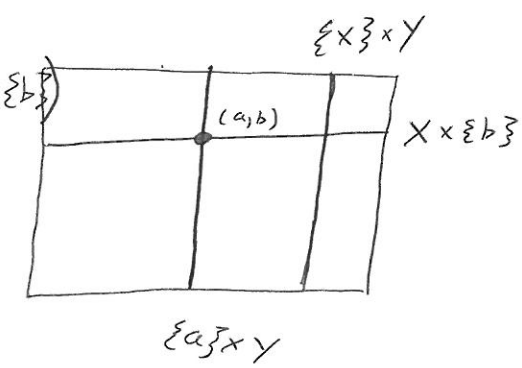
Pf Exercise

Thus, $X \times \{b\}$ is connected for any $\{b\}$ and

$\{a\} \times Y$ is connected for any $\{a\}$.

Fix $a \in X$ and $b \in Y$

For $x \in X$ define $T_x = (\{x\} \times Y) \cup (\{X\} \times \{b\})$



Claim T_x is connected

Pf Since $\{x\} \times Y$ is connected, $X \times \{b\}$ is connected and

$(a, b) \in T_x$, then by Prop 1 T_x is connected

Notice that $X \times Y = \bigcup_{x \in X} T_x$, and $(a,b) \in \bigcap_{x \in X} T_x = X \times \{b\}$

Since T_x is connected for every x, then by

Prop 1 $X \times Y$ is connected. $\square$

Big Th^m $\mathbb{R}$ with the standard topology is connected.

Pf Suppose to form a contradiction $\mathbb{R} = A \cup B$.

Let $a \in A$ and $b \in B$. WLOG assume $a < b$.

Let $A_0 = A \cap [a,b]$

$B_0 = B \cap [a,b]$

Claim $[a,b] = A_0 \,|\, B_0$

Pf $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$

A_0 and B_0 are open in $[a,b]$

$A_0 \cup B_0 = [a,b]$

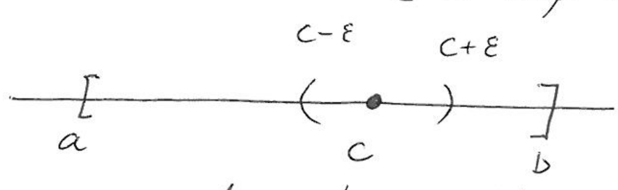
$A_0 \cap B_0 = \emptyset$.

Let $c = \sup A_0$. (c is the least element of $\mathbb{R}$ that is greater than or equal to all elements of A_0.)

Case 1: If $c \in B_0$, then $c \neq a$.

Since B_0 is open, $\exists\ \varepsilon > 0$ s.t. $(c-\varepsilon, c+\varepsilon) \cap [a,b] \subset B_0$.

Hence, $c - \varepsilon < c$ and $c - \varepsilon \geq a_0$ for every $a_0 \in A_0$ which contradicts $c = \sup A_0$



Case 2: If $c \in A_0$ then $c \neq b$.

Since A_0 is open $\exists\ \varepsilon > 0$ s.t. $(c-\varepsilon, c+\varepsilon) \cap [a,b] \subset A_0$

Hence $c + \frac{\varepsilon}{2} > c$ and $c + \frac{\varepsilon}{2} \in A_0$ which contradicts $c = \sup A_0$. $\square$