Topology Day 10

Outline

- Infinite Products
- Metric Spaces

Recall: Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of top. spaces. There are two important topologies on $\prod_{\alpha \in J} X_\alpha = \{\vec{x} : J \to \bigcup_{\alpha \in J} X_\alpha \mid \vec{x}(\beta) \in X_\beta \text{ for all } \beta \in J\}$ given by basis

$$\mathcal{B}_{box} = \left\{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ is open in } X_\alpha \right\}$$

$$\mathcal{B}_{prod} = \left\{ \prod_{\alpha \in J} U_\alpha \;\middle|\; \begin{array}{l} U_\alpha \text{ is open in } X_\alpha \text{ and} \\ U_\alpha = X_\alpha \text{ for all but} \\ \text{finitly many } \alpha. \end{array} \right\}$$

We saw last time that there exists products such that the box top. is strictly finer than the product top.

<u>Prop</u> Let $f : Y \to \prod_{\alpha \in J} X_\alpha$ where $\prod X_\alpha$ has the prod. top. Then f is cont. iff each component $f_\beta = \pi_\beta \circ f$ is continuous as a map $f_\beta : Y \to X_\beta$.

(Recall $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is a <u>projection map</u>

$$\pi_\beta(\vec{x}) = \vec{x}(\beta).$$

Pf] $\Rightarrow$] Claim $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is continuous.

Let $U_\beta \subset X_\beta$ be open.

$\pi_\beta^{-1}(U_\beta)$ is a sub-basis element of the prod. top.

Hence, $\pi_\beta^{-1}(U_\beta)$ is open.

Since π_β is cont. and f is cont. by hyp., then

$f_\beta = \pi_\beta \circ f$ is cont. since composition of continuous maps is cont. $\square$

$\Leftarrow$] Suppose each $f_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is cont.

~~W.T.S. f~~ WTS f is cont.

As noted in class, it suffices to show f^{-1} of ~~as~~ every subbasis element is open.

Recall $S = \{ \pi_\beta^{-1}(U_\beta) \mid \beta \in J \text{ and } U_\beta \subset X_\beta \text{ an openset} \}$ is a sub basis.

Examine $f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \circ f)^{-1}(U_\beta)$

$= f_\beta^{-1}(U_\beta)$ open by hyp.

Hence, the inverse image of any sub basis element is open. $\square$

Ex] Suppose $\prod_{i=1}^{\infty} \mathbb{R}$ has the box top. Let $f : \mathbb{R} \to \prod_{i=1}^{\infty} \mathbb{R}$ s.t. $f(t) = (t, t, t, \dots)$.

Since $f_i : \mathbb{R} \to \mathbb{R}$ is the identity, then it is continuous.

Note $B = \prod_{i=1}^{\infty} \left(\frac{-1}{i}, \frac{1}{i}\right)$ is open in $\prod_{i=1}^{\infty} \mathbb{R}$ with the box topology.

Examine $f^{-1}(B) = \{t \in \mathbb{R} \mid t \in (-1,1) \text{ and } t \in (\frac{-1}{2}, \frac{1}{2}) \text{ and } t \in (\frac{-1}{3}, \frac{1}{3}) \text{ and} \dots\}$.

Hence $f^{-1}(B) = \{0\}$. So, f is <u>not</u> cont.

This is a counter example to the previous prop. in the case where $\prod_{\alpha \in J} X_\alpha$ has the box top.

<u>Metric Spaces</u>

<u>Def</u> A <u>metric</u> on a set X is a function $d: X \times X \to \mathbb{R}$ s.t.

1) $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ iff $x = y$.
2) $d(x,y) = d(y,x)$
3) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \in X$

The pair (X, d) is a <u>metric space</u>.

<u>Def</u> The <u>metric topology</u> on a metric space X is given by the following basis

$$\mathcal{B} = \{B_\varepsilon(x) \mid \text{for all } \varepsilon > 0 \text{ and } x \in X\}$$

$$(B_\varepsilon(x) = \{y \in X \mid d(x,y) \leq \varepsilon\}.)$$

Exercise: Prove this is a basis.

As a consiquence of our lemmas regarding bases

$\mathcal{U} \subset X$ is open iff $\forall x \in \mathcal{U}\ \exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset \mathcal{U}$.

Ex| $X = \mathbb{R}$ with $d: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by $d(x,y) = |x-y|$
is a metric space. The coresponding metric
topology is identical to the standard topology.

Ex| Let X be any set, $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ is a metric.

The resulting metric topology is the discrete
topology.

Def| A top. space (X, τ) is <u>metrizable</u> if $\exists$ a
metric d on X s.t. the metric topology equals τ.

Def| A metric space (X, d) is <u>bounded</u> if there exists
$M > 0$ s.t. $d(x,y) \leq M$ for all $x, y \in X$.

Prop| Given a metric space (X, d) define $\bar{d}(x,y) = \min(d(x,y), 1)$.
Then $(X, \bar{d})$ is a bounded metric space and the metric topologies
for $(X, \bar{d})$ and (X, d) are the same.

Pf| Claim: $\bar{d}$ is a metric

Pf| Exercise.

Note that $\mathcal{B}_1 = \{ B_\varepsilon(x) \mid 1 > \varepsilon > 0$ and $x \in X \}$ is
a basis for the metric topology on (X, d)

Similarly $\mathcal{B}_1$ is a basis for the metric topology on $(X, \bar{d})$.
Hence, the metric topology on (X, d) is identical to the metric topology on $(X, \bar{d})$.

<u>Motivating Question</u>

How do we define a metric on $\prod_{i=1}^{\infty} \mathbb{R}$?

- can't use $\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots}$ (sum may not converge)
- can't use $\sup_{n \geq 1} \{|x_n - y_n|\}$ (since may be infinite).

Let $\bar{d}$ be the metric on $\mathbb{R}$ s.t. $\bar{d}(x,y) = \min\{|x-y|, 1\}$
Define $\bar{d}$ a metric on $\prod_{i=1}^{\infty} \mathbb{R}$ by

$$\bar{d}(\vec{x}, \vec{y}) = \sup_{n \geq 1} \{\bar{d}(x_n, y_n)\}$$

This is the <u>uniform metric</u> on $\prod_{i=1}^{\infty} \mathbb{R}$. (or $\prod_{\alpha \in J} \mathbb{R}$)

<u>Thm</u> On $\prod_{\alpha \in J} \mathbb{R}$ the box topology is finer than the uniform topology and the uniform topology is finer than the box top. <u>and</u> all are different if J is infinite.

<u>Pf</u> We will show that if J is infinite, then the box top. is strictly finer than the uniform top. (See Thm 20.4 for rest). Let $B_\varepsilon(\vec{x})$ be a basis element for the uniform top.

Let $\vec{y} \in B_\varepsilon(\vec{x})$ and let $\delta = \varepsilon - \bar{d}(\vec{x}, \vec{y}) > 0$.

Define $U = \prod_{\alpha \in J} (\vec{y}(\alpha) - \frac{1}{2}\delta, \vec{y}(\alpha) + \frac{1}{2}\delta)$

- Note that $\mathcal{U}$ is a basis element of the box topology and $\vec{y} \in \mathcal{U}$.
- If we show that $\mathcal{U} \subset B_\varepsilon(\vec{x})$, then by 13.3 we have showed the box top. is finer than the uniform top.
- Let $z \in \mathcal{U}$, then $\bar{d}(\vec{y}, \vec{z}) < \frac{\delta}{2}$.

Hence $\bar{d}(\vec{x}, \vec{z}) \leq \bar{d}(x, y) + d(\vec{y}, \vec{z})$

$$\leq \varepsilon - \delta + \tfrac{1}{2}\delta$$

$$\leq \varepsilon - \tfrac{\delta}{2}.$$

So $\vec{z} \in B_\varepsilon(\vec{x})$. Hence $\mathcal{U} \subset B_\varepsilon(\vec{x})$. $\square$

So, box is finer than uniform.

To show strictly finer examine

$\prod_{i=1}^{\infty} \left(\frac{1}{i}, \frac{-1}{i}\right)$ open in $\prod_{i=1}^{\infty} \mathbb{R}$ with box top.