Math 550A, Homework 6

Compactness

Due in class, Thursday, 3/27

Reading §26–28

Exercises (to do on your own)

1. Does every topological space have a finite cover?
2. Prove that the unit n-sphere S^n is compact.
3. Prove that $\mathbb{R}$ with the finite complement topology is compact.

Problems (to turn in)

1. Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ if $n > 1$ (Hint: consider what happens if you delete a point from each space).
2. Munkres §26, exercise 8. This is an example of a "closed graph theorem." (You may assume exercise §26.7, as the hint suggests. Recall that a closed map sends closed sets to closed sets.)
3. Munkres §28, exercise 7, parts a) and b) only.
4. Let $Z = \mathbb{R} \cup \{*\}$, where $\{*\}$ is a one-point set (that is not a subset of $\mathbb{R}$). Put a topology on Z using the basis consisting of all open intervals in $\mathbb{R}$, together with all sets of the form $(a, \infty) \cup \{*\} \cup (-\infty, -a)$ for $a > 0$. (Think of "gluing" the point $*$ in such a way as to join $-\infty$ and ∞.) Prove that S^1 is homeomorphic to Z by completing the following steps.

   (a) Recall that $x \times y \in \mathbb{R}^2$ belongs to S^1 iff $x^2 + y^2 = 1$. Define $f : S^1 \to Z$ by

   $$f(x, y) = \begin{cases} \frac{x}{1-y}, & \text{if } y \neq 1 \\ *, & \text{if } y = 1. \end{cases}$$

   Prove that f is bijective. (Hint: prove that if $y \neq 1$, then there exists a unique line in $\mathbb{R}^2$ containing the point 0×1 and the point $x \times y$. The value of $f(x, y)$ is where this line crosses the x-axis. f is called *stereographic projection.*)

(b) Prove that f is continuous by showing the inverse image of any basic open set in Z is open in S^1. You may draw pictures to aid in your argument.

(c) Use a trick from class to automatically conclude that f is a homeomorphism. Be sure to justify all your steps.

Remark: An analogous argument shows that S^n is homeomorphic to $\mathbb{R}^n$ glued to a single "point at infinity" for any $n \geq 1$.