Math 550A, Homework 5

Midterm 1, Metric spaces and Connectedness

Due in class, Tuesday, 3/18

Reading Read §20 and §23 of Munkres.

Exercises (to be done on your own)

1. Prove that the collection of all ϵ-balls in a metric space actually forms a basis for a topology.
2. If (X, d) is a metric space, verify that $\overline{d}(x, y) = \min(d(x, y), 1)$ defines a metric on X.
3. Munkres §20, exercise 4.
4. Suppose that d and d' are metrics on a set X. d and d' are said to be *uniformly equivalent* if there exist positive real numbers a, b such that
$$ad(x, y) \leq d'(x, y) \leq bd(x, y)$$
for all $x, y \in X$.
   (a) Prove that if d and d' are uniformly equivalent, then they induce the same topology on X. (You may use results from Munkres.)
   (b) Prove that on $\mathbb{R}^n$, the Euclidean metric d, the square metric ρ, the "taxicab metric"
$$\tau(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \ldots + |x_n - y_n|.$$
   are all uniformly equivalent to each other.
5. Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ if $n > 1$ (Hint: consider what happens if you delete a point from each space).
6. Given a topological space X, define a relation $\sim$ on X by setting $x \sim y$ if there exists a connected subspace of X containing both x and y.
   (a) Verify that $\sim$ is an equivalence relation. (See §3 for the definition of equivalence relation.)

(b) A *component* of X is an equivalence class for $\sim$ (again, see §3). Prove that the components of X are connected, disjoint subspaces whose union is X.

(c) Prove that any connected subspace of X is contained in some component.

Problems (to turn in) Please turn in your midterm 1 along with solutions to problems 2,3,4 and 5. If you received a perfect score on one of these problems, then you do not need to rewrite that problem. You will be unable to earn partial credit on this homework, any answer that is not totally correct will receive zero points.