Final Exam

Math 500, Fall 2011

INSTRUCTIONS:

- This exam is due in my mailbox or my inbox by noon on December 20th.
- You may use your textbook (Munkres), your own homework assignments, your class notes, the first midterm, and the second midterm but NO OTHER SOURCES. No other books, no discussing with other people, no internet sources.
- Abide by Penn's Code of Academic Integrity. If you have any questions or concerns, let me know.
- If you have questions about the exam or if you think there is a mistake, please email me (ryblair@math.upenn.edu).
- Write legibly, in COMPLETE SENTENCES, and explain your work carefully. Please write up your solutions very clearly and try to give an "appropriate" level of detail. If there is any doubt about how much detail to give, or about what you may quote without proof, just ask me by e-mail.
- You may quote results from class and from Munkres that we have discussed. (You don't need to re-prove anything we have already done.)
- Typing the solutions is preferred, but only necessary if you have been asked to do so.
- Good luck!

Name ______________________________

Total Score __________ (/ 60 points)

1. *(10 points)* Show that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all x, then g is surjective.

2. *(10 points)* Mark each of the following statements true or false. If you mark a statement as false provide a counter example. No additional justification is necessary.

   (a) If E is homeomorphic to B, then any covering map $p : E \to B$ is injective.

   (b) If the product of spaces under the product topology is regular, then each of the spaces is regular.

   (c) Given any continuous map $f : S^1 \times S^1 \to \mathbb{R}^2$ there exists some point $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$ where $-x$ denotes the antipodal point to x on S^1.

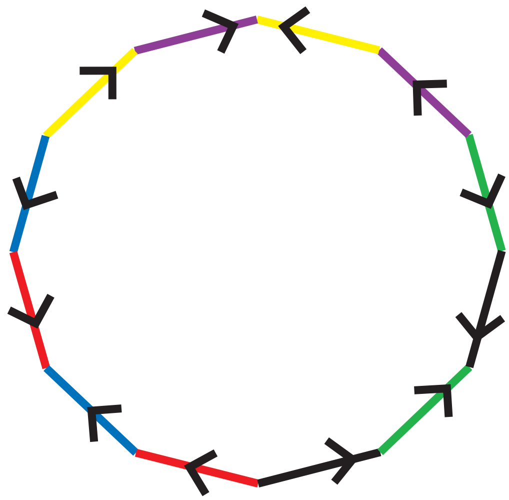
   (d) If X is a bounded metric space, then X is not homeomorphic to an unbounded metric space.

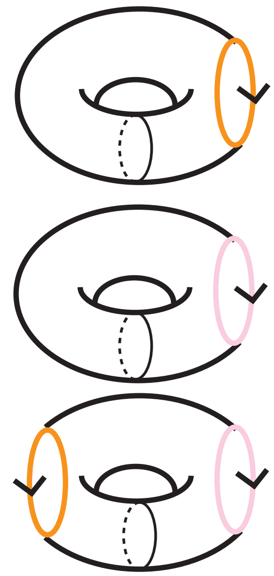
3. *(10 points)* Give the set $\mathbb{R}^3$ the product topology on $\mathbb{R}^2 \times \mathbb{R}$ where $\mathbb{R}^2$ has the ordered topology and $\mathbb{R}$ has the finite complement topology. What is a basis for the subspace topology on $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$? Justify your answer.

4. *(10 points)* Show if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

5. *(10 points)* Let $f : X \to Y$ and $g : X \to Y$ be continuous functions and assume Y is Hausdorff. Show $\{x \in X | f(x) = g(x)\}$ is closed.

6. *(10 points)* Using a "cut-and-paste" argument similar to the one used in class to demonstrate that $\mathbb{RP}^2$ is homeomorphic to the disk glued to the Mobius band along their S^1 boundary, show that the following quotient spaces are homeomorphic. (i.e. the disk with regular dodecagon boundary with the following side identifications is homeomorphic to the disjoint union of two once punctured tori and one twice punctured tori under the given boundary identifications.) Hint: Use colors and pay attention to the orientations on the identified edges.





b) Based on our experience finding the fundamental group of the genus two surface, guess a presentation of the fundamental group of the above space. You need not justify your answer.

7. *(10 points)*

a) Read the definition of deformation retract on page 361 and the definition of a wedge of circles on page 434. Use Theorem 58.3 and the version of the Seifert-van Kampen theorem given in class to show the fundamental group of the wedge of n circles is isomorphic to the free group on n generators, i.e. $\pi_1(\bigvee_{i=1}^n S^1, x_0) \cong F_n$. DO NOT JUST COPY THE PROOF IN THE BOOK.

The Seifert-van Kampen Theorem:

Suppose X is a path connected topological space such that $X = U \cup V$ where U and V are path connected open sets and $U \cap V$ is path connected. Let $\phi_1 : U \cap V \to U$ and $\phi_2 : U \cap V \to V$ be the natural inclusion maps and let $x_0 \in U \cap V$. Then

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$$

where the above amalgamation is via the induced maps $(\phi_1)_*$ and $(\phi_2)_*$.

b) Use part a) and Theorem 58.3 to find the fundamental group of the compliment of n distinct lines through the origin in $\mathbb{R}^3$. Justify your answer.