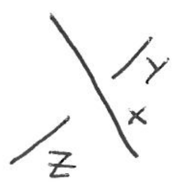
Knot theory Day 6

Outline

- New H.W up this afternoon

- mod p lables
- the linear algebra of lables.

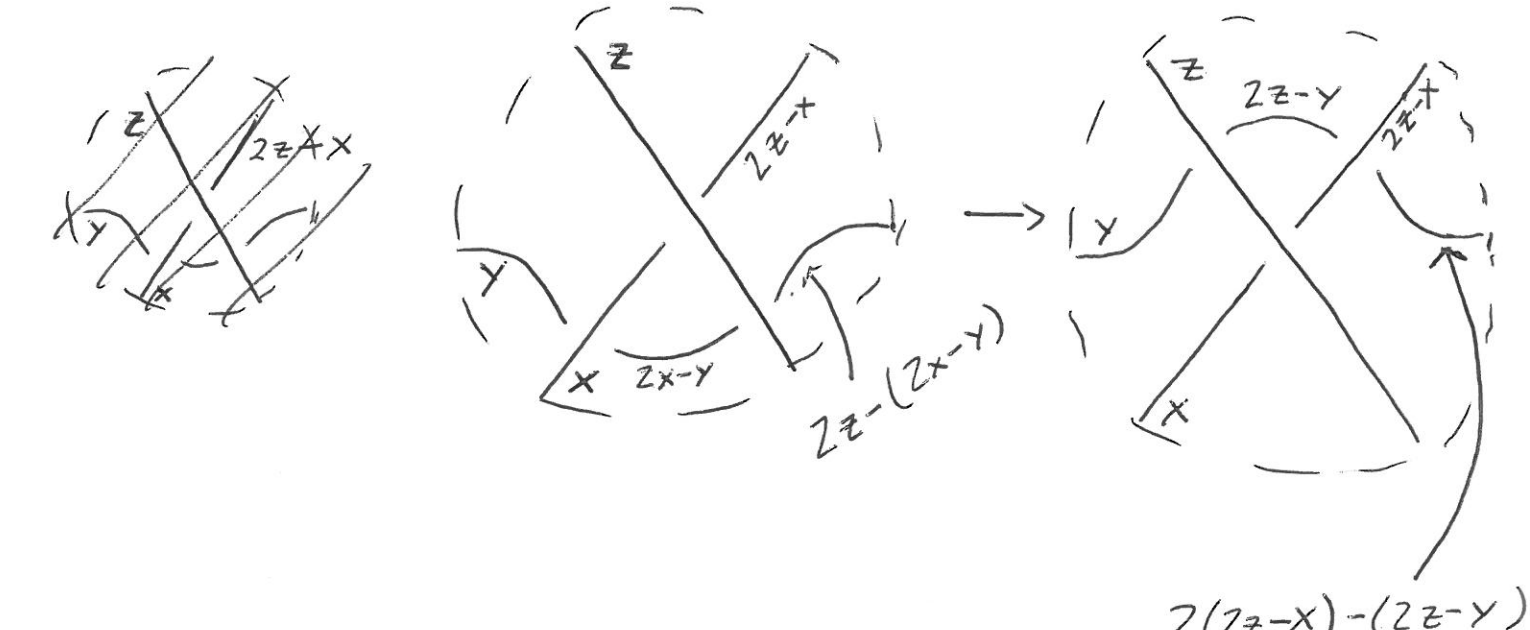
<u>Def</u> A knot diagram can be labled mod p if each edge can be labled with an integer from 0 to $p-1$ s.t. at each crossing $2x-y-z \equiv 0 \bmod p$ where x is the lable on the overcrossing and y and z are the two other lables incident to the crossing. i.e.



Note: Under this definition every knot has p distinct mod p lables.

<u>Def</u> A knot diagram has a <u>non trivial</u> labling mod p if the diagram can be labled mod p s.t. at least two distinct lables are used.

Claim: The following Reidermeister move induces a unique ~~p~~ mod p labeling on the diagram



Must verify that $2(2z-x)-(2z-y) \equiv 2z-(2x-y) \bmod p.$

$$2(2z-x)-(2z-y) = 4z-2x-2z+y$$
$$= 2z-(2x-y). \checkmark$$

Hence, a labeling of the diagram before this Reidermeister move induces a unique labeling of the diagram after the move.

Exercise | Verify this for all remaining Reidermeister moves.

Thm | The number of mod p lables of any diagram is a knot invariant.

Example

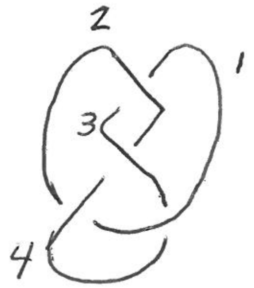


Fig. 8 knot has a non-trivial labeling mod 5.

$$\begin{cases} 2b - a - x \equiv 0 \bmod 5 \\ 2a - b - x \equiv 0 \bmod 5 \end{cases}$$

$$2x - b - a \equiv 0 \bmod 5$$

① $x \equiv 2b - a$

$$2a - b \equiv 2b - a$$

$$3a \equiv 3b$$

$$a \equiv b \quad \leftarrow \text{working mod a prime}$$

So, the trefoil has only trivial labelings mod 5.

Hence, mod 5 lables distinguish the figure 8 knot and the trefoil.

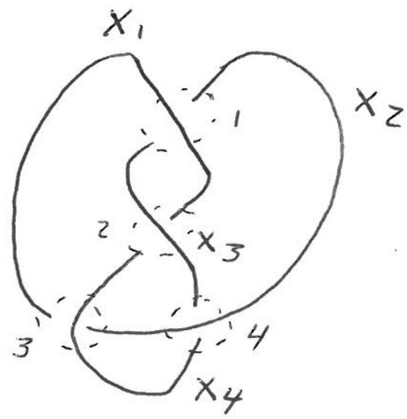
To show that the number of mod p labelings is a knot invariant we must show that any labeling of a knot before any Reidermeister move induces a <u>unique</u> labeling after the Reidermeisler move.

Recall that the integers mod p where p is prime is a <u>field</u> where as the integers mod p where p is not prime is a ring.

We can use linear algebra to calculate the number of ~~p colorings of~~ mod p labelings of a knot diagram.

(for simplicity, we will assume p is a prime so that we are working with vector spaces instead of modules)

Step 1: Assign to each arc of the diagram a variable x_i.



Step 2: Each crossing induces a linear equation over the integers mod p.

1) $2x_1 - x_2 - x_3 \equiv 0$

2) $2x_3 - x_1 - x_4 \equiv 0$

3) $2x_4 - x_2 - x_1 \equiv 0$

4) $2x_2 - x_3 - x_4 \equiv 0$

Step 3: We produce a matrix equation mod p.

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A\vec{x} = \vec{0}$$

Step 4: ~~The~~ From linear algebra over finite fields, the number of solutions to this matrix eq. mod p is $p^{(\text{nullity of } A)}$. (Recall nullity of A is the dimention of the null space of A).

Step 5: Find the nullity of A using row reduction mod p.

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -1 \end{bmatrix} \xrightarrow{R_1+R_2+R_3 \to R_1} \begin{bmatrix} 0 & 2 & -1 & \\ -1 & 0 & & \\ & & & \\ & & & \end{bmatrix}$$

$$\xrightarrow[R3-R_2 \to R3]{R1+2R_2 \to R1} \begin{bmatrix} 0 & -1 & 3 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -2 & 3 \\ 0 & 2 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow[R_4+2R_1 \to R_4]{R_3-R_1 \to R_3} \begin{bmatrix} 0 & -1 & 3 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -5 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 4 & 0 \\ 4 & 0 & 2 & 4 \\ 4 & 4 & 0 & 2 \\ 0 & 2 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 4 & 0 \\ 0 & 2 & 4 & 4 \\ 0 & \phi & 2 & 2 \\ 0 & 2 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 4 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so, nullity of A is 2.

Hence, the figure 8 knot has 5^2 mod 5 lables.