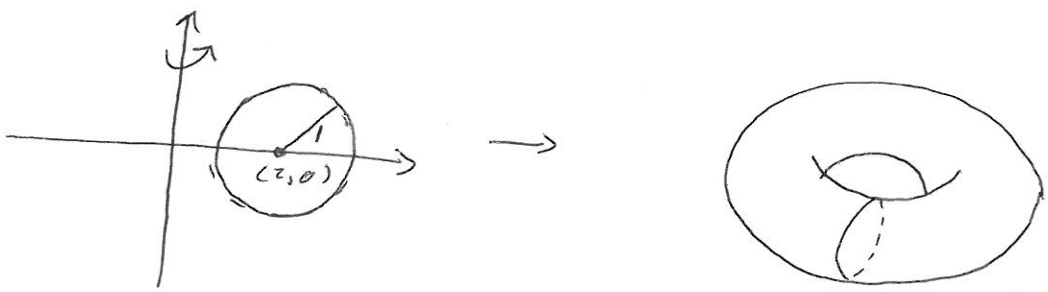
Knot Theory Day 20

Exam
40–50 A
30–39 B
25–29 C
Ave 37
Median 35

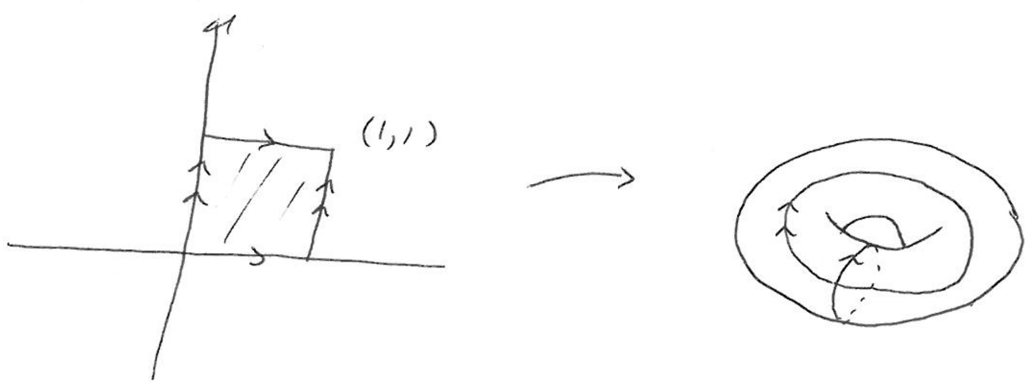
Outline
- Torus knots
- Satelite knots
- Hyperbolic knots.

Torus knots

The standard torus in $\mathbb{R}^3$ is a surface of revolution

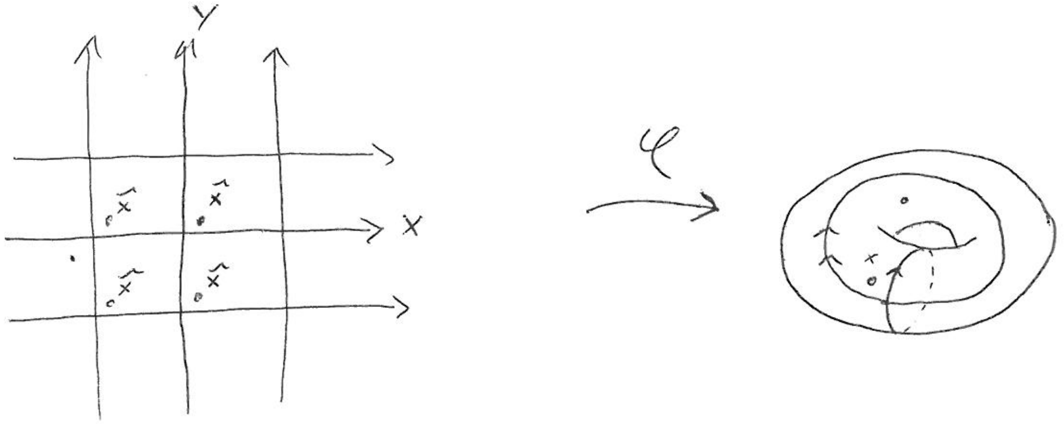


Recall: The torus can be obtained from the square by Identifying edges in the following way.



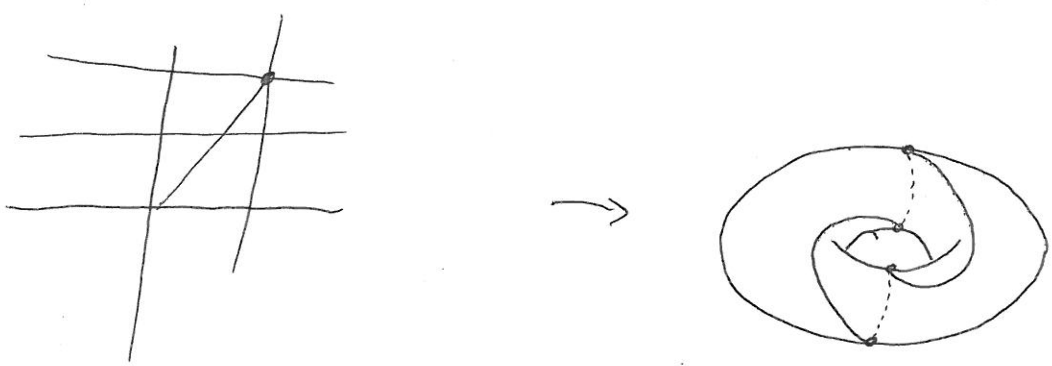
This gives rise to a natural map called a covering map $\varphi : \mathbb{R}^2 \to$ Torus.

s.t. $\varphi((x,y)) = (x \bmod 1, y \bmod 1)$

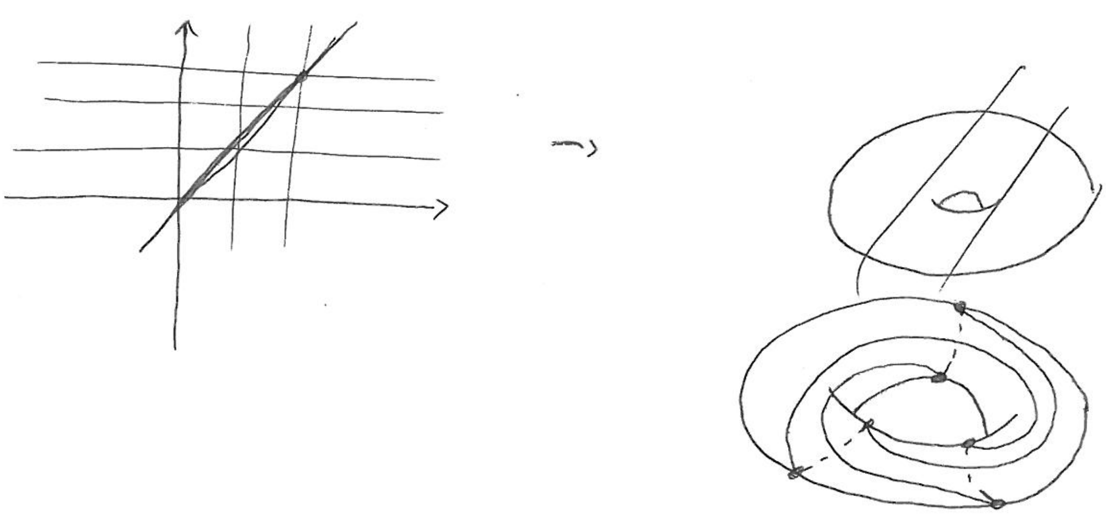


under this map
- vertical lines get mapped to the meridian
- horz. lines get mapped to the longitude
- lines through the orgin of rational slope get mapped to simple closed curves in the Torus

Ex | Line with slope $2/1$ gets mapped to



with slope $3/2$

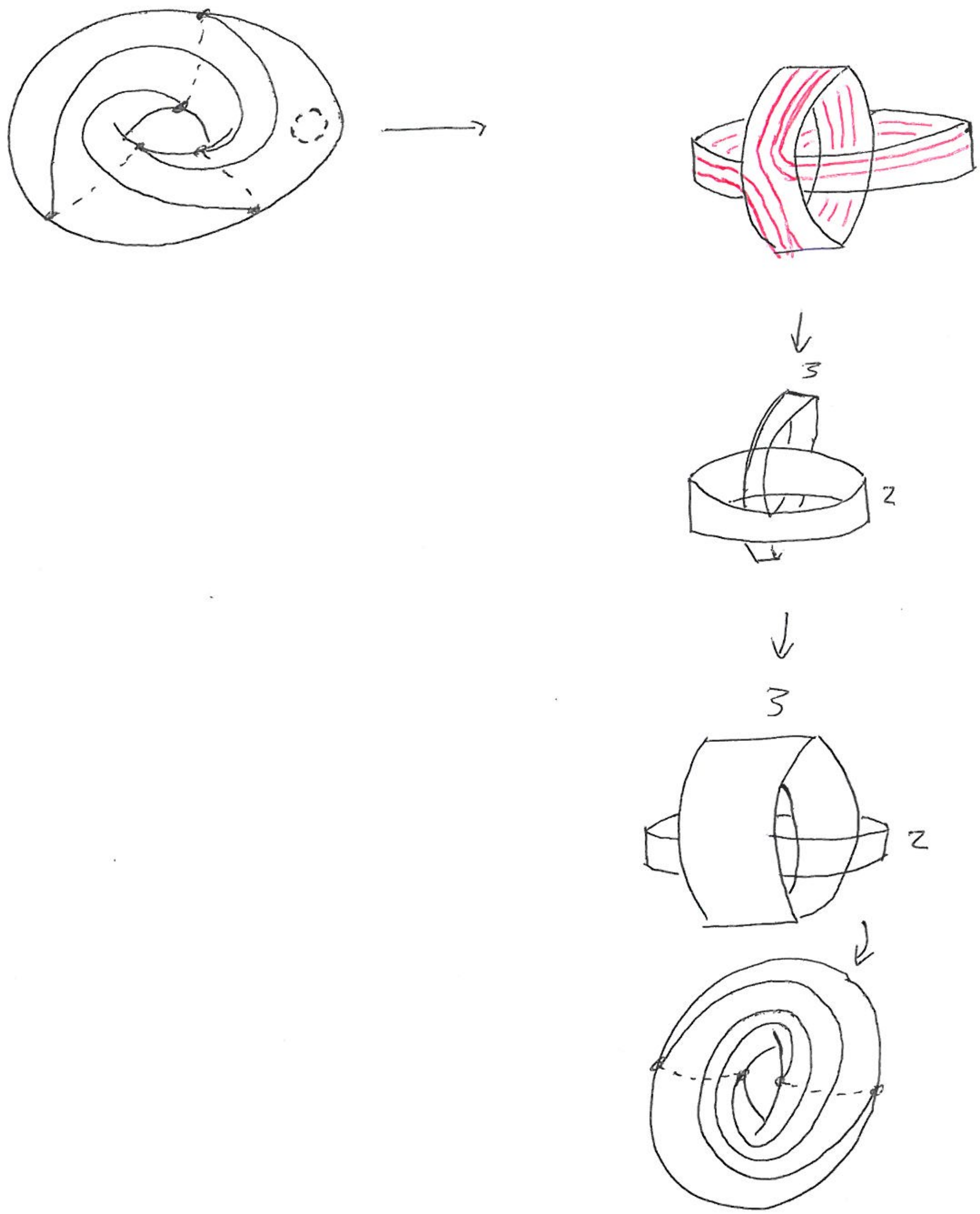


Fact | Every isotopy class of non-trivial simple closed curve in T^2 ~~is~~ contains the image of some line through the origin of rational slope and every line of rational slope gets mapped to a distinct isotopy class.

Hence, (up to isotopy) every simple closed curve on the torus is the image of a line through the origin of slope p/q. So, we call a curve on T^2 a p/q curve.

Def | A torus knot is any knot that is ambient isotopic to a simple closed curve on the standard torus embedded in $\mathbb{R}^3$.
We call a knot isotopic to a p/q curve on the standard torus a p/q torus knot.

Fact: A p/q torus knot is equivalent to a q/p torus knot



Facts about torus knots

- The crossing number of a p/q torus knot is
  $\min((p-1)q, (q-1)p)$
- $c(K_1 \# K_2) = c(K_1) + c(K_2)$ if K_1 and K_2 are torus knots (Yuanan Diao)
- Applying Seiferts algorythm to a standard projection of a torus knot yields a minimal genus Seifert Surface.

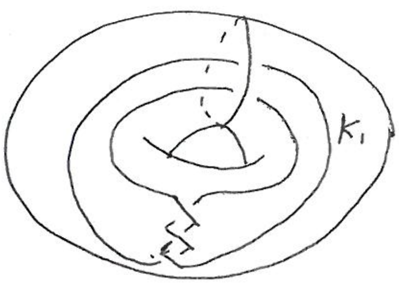
Satellite Knots

Let K_1 be a knot cotained inside of a solid torus V_1. Let K_2 be a knot in $\mathbb{R}^3$. Let V_2 be a closed regular nbh of K_2. Replace $V_2 \subset \mathbb{R}^3$ with V_1 s.t. a meridian disk of V_1 is mapped to a meridian disk of V_2. Let K_3 be the image of K_1 under this replacement.

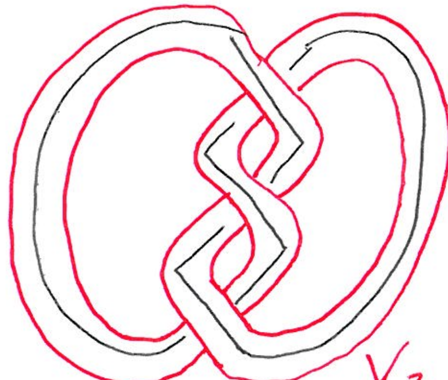
K_3 is a <u>satellite knot</u>

K_2 is a <u>companion knot</u>

K_1 is the <u>pattern knot</u>

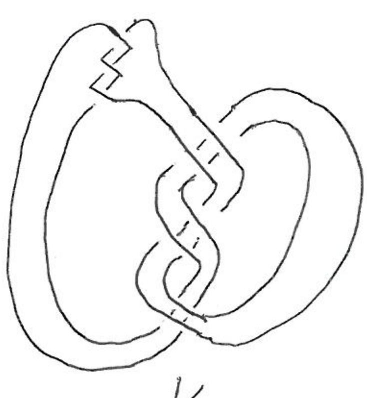


V_1



K_2

V_2



K_3

Knot Theory Day 21

- New HW up.
- "Curved spaces" Jeff weeks

Outline
- Satellite knots
- Hyperbolic knots

- Recall

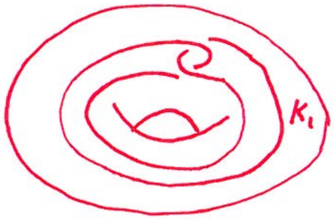
- Torus knots are knots equivalent to a simple closed curve on the standary torus in $\mathbb{R}^3$.

- We can describle every torus knot as the image of a line $y = p/q\, x$ under the map $\varphi: \mathbb{R}^2 \to S^1 \times S^1$ via $\varphi((x,y)) = (e^{2\pi x i}, e^{2\pi y i})$.

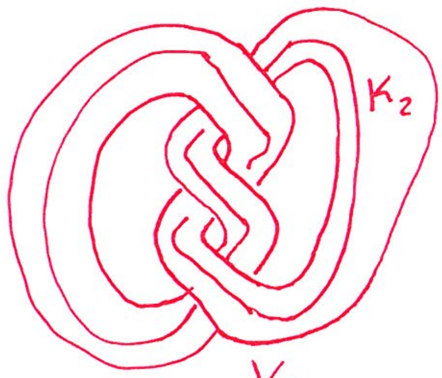
Satellite knots

Suppose K_1 is a knot in a solid torus V_1. Let K_2 be a knot in $\mathbb{R}^3$ s.t. the closed regular nbh of K_2 is V_2. Let $f: V_1 \to V_2$ be ~~a map~~ a homeomorphism of solid tori. Let K_3 be the knot $f(K_1)$.

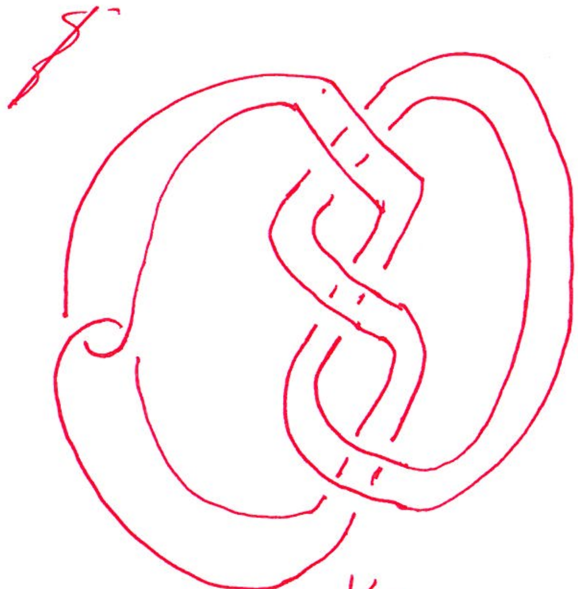
K_1 is the pattern knot.
K_2 is the companion knot.
K_3 is the satellite knot.



V_1

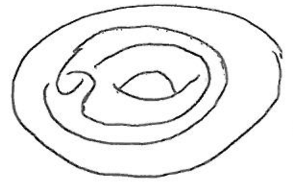


V_2



K_3

Def | A <u>whitehead double</u> is any satelite knot with pattern



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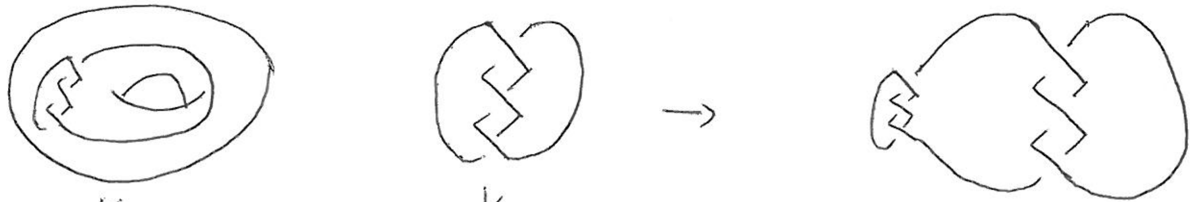
Def | A <u>cable knot</u> is any satellite knot with ~~companion~~ pattern a standard embedding of a torus knot.

Fact | Every connected sum $K_1 \# K_2$ is a satellite knot with pattern K_1 and companion K_2.



K_1 $\quad$ K_2 $\quad$ K_3.

Thm | (Schubert) If K_3 is satellite with companion K_2 then $\beta(K_3) \geq b\beta(K_2)$ where b = minimal number of intersections between the pattern and a meridian disk for V_1.

Open Question | If K_3 is satellite with companion K_2 then $c(K_3) \geq c(K_2)$.

Thm (Thurston)

Every knot fits into exactly one of the following three categories.

1) Torus knots
2) Satellite knots
3) Hyperbolic knots.

Thm (Menasco)

Let K be a prime alternating knot that is not a $2/n$ torus knot, then K is hyperbolic.

Ex

is hyperbolic

Def A hyperbolic knot is a knot ~~with~~ s.t. $S^3 - K$ can be given a complete metric of constant curvature -1.

What does that mean?

In 2D

positive curvature

zero curvature

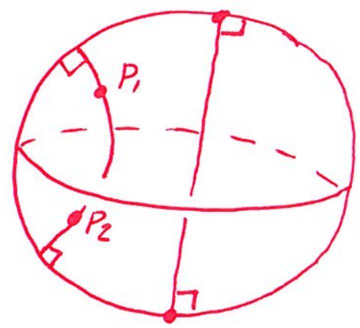
Negative curvature

Note| Download Jeff weeks program.

The simplest Hyperbolic 3-manifold.

$$H^3 = \{ (x, y, z) \mid x^2 + y^2 + z^2 < 1 \}$$

with a metric such that the geodesics in H^3 are straight line segments or sub arcs of circles that meet ∂H^2 in right angles.



If ω is a geodesic from P_1 to P_2, points in H^3, then the distance from P_1 to P_2 is

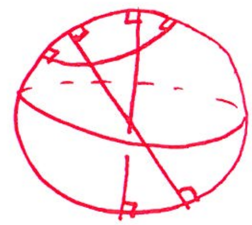
$$d(P_1, P_2) = \int_\omega \frac{2\,ds}{1 - r^2}$$

where we integrate wrt arc length in euclidian metric and r is the distance to the orgin.

So if $\omega(t) = (x(t), y(t))$

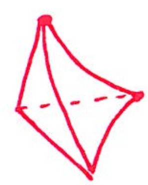
$$d(\omega(t_0), \omega(t_1)) = \int_{t_0}^{t_1} \frac{2\sqrt{(x'(t))^2 + (y'(t))^2}}{1 - \left((x(t))^2 + (y(t))^2\right)}\,dt$$

A *hyperbolic triangle* in $\mathbb{H}^3$ has edges consisting of sub arcs of geodesics.



The sum of the angles of a hyperbolic triangle is always less than 180°.

A *hyperbolic tetrahedra* has edges consisting of sub arcs of geodesics and faces consisting of portions of spheres that meet $\partial\mathbb{H}^3$ in right angles.



All hyperbolic tetrahedra have finite volume!

Crazy!

Def

A *hyperbolic 3-manifold* is the union of finitly many hyperbolic tetrahedra glued together allong faces in such a way that there is no metric distortion.

Def A *hyperbolic knot* is a knot s.t. $S^3 - K$ is a hyperbolic 3-manifold.