Math 123: From Parametric Curves to Polar Coordinates

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Outline

Parametric Curves

Definition

A *parametric curve* in the xy-plane is given by $x = f(t)$ and $y = g(t)$ for $t \in [a, b]$.

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Theorem

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Example: Find the length of one arch of the cycloid.

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Example: Derive these conversion rules.
Example: Sketch the graph of $r = cos(2\theta)$.

Derivatives of Parametric Curves

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So, if $r = f(\theta)$ is a curve in polar coordinates, then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} sin(\theta) + rcos(\theta)}{\frac{dr}{d\theta} cos(\theta) - rsin(\theta)}$$

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Example: Find the slope of the curve $r = cos(2\theta)$ at $\theta = \frac{\pi}{4}$.
Example: Find the points on the curve $r = e^{\theta}$ where the tangent line is horizontal or vertical.

Area in Polar Coordinates

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Example: Find the area enclosed by one leaf of $r = cos(2\theta)$.