

Optimal Lot Sizing Under Carload Discount Schedules

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Abstract

In order to encourage customers to buy more of an item, suppliers often offer discount schedules, most common ones being all-units and incremental discounts. The *carload discounts* schedule is one of

the simplest schedules that are commonly used in practice. In this paper, an optimal lot sizing procedure is developed for this model. It turns out that the procedure is extremely simple, contrary to the expectations reported in the literature.

1 Introduction

In his seminal paper, Harris (1913) assumed linear procurement costs in the formulation of the EOQ problem. But when sellers offer quantity discounts for large orders, nonlinearities arise in the cost function. The effect of quantity discounts on purchase quantities has been investigated at least since Whitin (1953). The discount schedules, that have been extensively discussed in the literature are *all units discounts* (Churchman et al. 1957) and *incremental discounts* (Hadley and Whitin 1963) [also referred to as “n-block tariffs” in Dolan (1987).] Most of the work since then deals with these two most common types under the assumption that seller’s discount price breakpoints are given. For a review of these discount schedules under various scenarios and view points, see Benton and Park (1996).

A simple schedule that is used in practice is the *carload discounts* schedule described by Nahmias (2001):

A carload consists of M units. The supplier charges a constant c per unit up until you have paid for the cost of a full carload, at which point, there is no charge for the remaining units in that carload. Once the first carload is full, you again pay c per unit until the second carload is full and so forth.

This simple discounting schedule is analogous to the commonly known “cheaper by the dozen” discounting that is used in the retail industry. In essence, the carload discount schedule offers M units for the price of $N(< M)$ units, making it intuitive to order the full “carload”, M units, if the customer’s order quantity is close to M . This type of discount schedule is commonly used in practice. In a paper that discusses the managerial issues and research opportunities in quantity discounts, Dolan (1987) states four examples, three of which involve carload discounts:

... we have all seen advertisements like the following:

- “Buy the series and get one show free” from the theater group.
- “Join the family shoe plan. Buy 10 pairs and get one pair free” at the shoe store.
- “Buy a case of assorted bottles and get 20% off” at the wine store.

Suppliers may be inclined to offer such a discount schedule due to the specifics of their manufacturing processes, transportation modes or raw materials used. From a manufacturing perspective, products are typically produced in fixed batch sizes, making it favorable to induce buyers to order in multiples of batch sizes. In addition, when suppliers are responsible for transportation, they would prefer full truck-load shipments. If a full trailer or sea container load is ordered, the transportation costs are reduced due to full utilization of delivery resources. Raw material-driven discounts are common in the paper and metal industries, where suppliers offer discounts if

the entire roll of paper or sheet of metal is utilized to fill an order. This may reduce scrap, lead time and floor space requirements, leading to additional capacity in the supplier's plant and saving on inventory holding and material handling costs. Some or all of these cost savings may be passed on to the customer in the form of a quantity discount.

In this paper we address the lot sizing problem under the carload quantity discount schedule. We analyze the resulting total cost function and show that the procedure to find the optimal order quantity amounts to the comparison of three candidate points.

The rest of the paper is organized as follows. Section 2 presents a general formulation for the quantity discount lot sizing problem. In Section 3 the lot sizing problem under carload discount schedules is defined and analyzed. Five results are proposed and proved in this section. These propositions and the analysis based on these, constitute the essence of the solution procedure presented. For illustration, a small numerical example is given.

2 General Formulation

Let us consider a discount schedule, in which the cost of procuring Q units is given by:

$$C_j(Q) = \alpha_j + \beta_j Q, \quad b_j \leq Q < b_{j+1}, \quad j = 0, 1, \dots, m, \quad (1)$$

where b_j 's are the *price breakpoints* and α_j, β_j are given constants.

Many commonly encountered discount schedules are special cases of this general expression. For instance, if α_j 's are set equal to zero for $j = 0, 1, \dots, m$,

and β_j 's are decreasing as j increases, the model reduces to the *all units discount* schedule. If, as β_j 's decrease, α_j 's are chosen in such a way that $C_j(b_{j+1}) = C_{j+1}(b_{j+1})$ for $j = 0, 1, \dots, m$, the model represents the *incremental discount* schedule. If α_j 's are taken as the number of "free units in the carload" multiplied by the index j , $j = 0, 1, \dots, m$, the *carload discount* schedule is obtained. In order to encourage orders to be in multiples of some number, which may be dictated by manufacturing or packaging considerations, β_j 's can be set equal to zero and α_j 's selected to be increasing for $j = 0, 1, \dots, m$, the general model becomes what may be called the *multiples discount* schedule. Finally, when α_j 's are set to zero, and β_j 's are all equal to a positive constant independent of j , the model reduces to the case without quantity discounts.

These various cases are summarized below where b_j , $j = 0, 1, \dots, m$, are the breakpoints and the constant $(M - N)$ is the number of free units in the carload.

Discount Schedule	α_j	$\beta_j \geq 0$
All units	$\alpha_j = 0$	$\beta_j > \beta_{j+1}$
Incremental	$\alpha_{j+1} = \alpha_j + (\beta_j - \beta_{j+1})b_{j+1}$, $\alpha_0 = 0$	$\beta_j > \beta_{j+1}$
Carload	$\alpha_j = -j\beta(M - N)$, $\alpha_0 = 0$	$\beta_j = \beta$
Multiples	$\alpha_j < \alpha_{j+1}$	$\beta_j = 0$
No Discount	$\alpha_j = 0$	$\beta_j = \beta$

In order to complete the formulation, let us define the following parameters:

d : Demand rate (units/unit time),

k : Order cost (\$/order),

h : Unit inventory holding cost (\$/unit/unit time).

Furthermore, assume that inventory holding cost solely depends on the cost of money tied up in inventories. That is,

$$h = i \times \frac{C_j(Q)}{Q}$$

where i is the *inventory carrying charge* (per unit time). Using the classical EOQ formulation, the total cost per unit time for an order of size Q is given by

$$TC_j(Q) = \frac{kd}{Q} + \frac{dC_j(Q)}{Q} + \frac{1}{2}hQ, \quad (2)$$
$$b_j \leq Q < b_{j+1}, \quad j = 0, 1, \dots, m.$$

Rearranging terms and replacing h by $i \times C_j(Q)/Q$ and $C_j(Q)$ by $\alpha_j + \beta_j Q$, we obtain

$$TC_j(Q) = \frac{(k + \alpha_j)d}{Q} + \frac{(2d\beta_j + i\alpha_j)}{2} + \frac{i\beta_j Q}{2}, \quad (3)$$
$$b_j \leq Q < b_{j+1}, \quad j = 0, 1, \dots, m.$$

Furthermore, The total cost curve $TC_j(Q)$ is said to be *realizable*, if $TC(Q) = TC_j(Q)$, i.e. Q is in the half interval $[b_j, b_{j+1})$.

Thus the total cost curve for all positive values of Q is given by consecutive realizable segments of $TC_j(Q)$ for $j = 0, 1, \dots, m$,

$$TC(Q) = TC_j(Q), \quad b_j \leq Q < b_{j+1}, \quad j = 0, 1, \dots, m. \quad (4)$$

In general, $TC(Q)$ is neither continuous nor differentiable everywhere.

3 Carload Discount Schedule

In this section the *carload discount schedule* will be formally defined. The form of the total cost function will be analyzed and based on this analysis, an optimal lot sizing procedure will be presented. A small numerical example concludes the section.

3.1 The Model

Nahmias's (1989) description of a carload discount schedule was given in Section 1. There is a unit price c for every unit ordered and a cost of $c \times N$ for any M units of the order. In other words, any M units ordered are only charged the price of $N < M$ units and $(M - N)$ units are given free of charge. The model is most applicable when one has the option of either buying by the unit or by the case (of M units) where it costs $\$c$ per unit and $\$Nc$ per case. This kind of pricing is commonly used in the retail industry.

The total cost function is depicted in Figure 1 where it is shown with bold lines. It has the following implication: the optimal order quantity can never occur at the quantities for which the cumulative purchasing cost remains constant (i.e. between $N + jM$ and $(j + 1)M$ for $j = 0, 1, \dots, m$) because one can always order a higher quantity (e.g. $(j + 1)M$) for the same cost. (Since the unit holding cost is assumed to be a function of only the purchase cost and that the purchase cost is zero for units between $N + jM$ and $(j + 1)M$, the holding cost is also zero.)

Thus, there is no loss of generality if we append the “saw teeth” (shown with the dashed lines in the figure) over the regions where optimal Q can

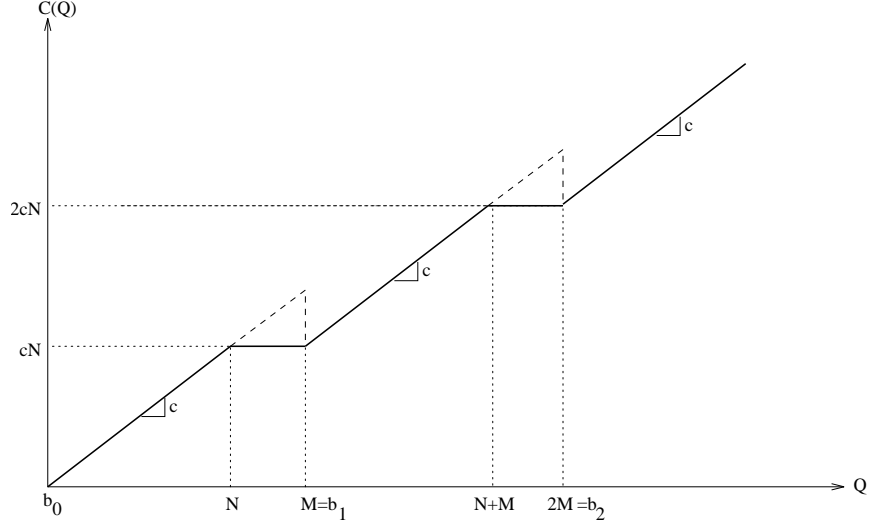


Figure 1: The cumulative purchase cost.

never occur, i.e. between $N + jM$ and $(j + 1)M$ for $j = 0, 1, \dots, m$. This latter form permits us to write the cumulative purchase cost function as,

$$C_j(Q) = cQ - jc(M - N), \quad b_j \leq Q < b_{j+1}, \quad j = 0, 1, \dots, m. \quad (5)$$

Clearly, this is a special case of the general form discussed in Section 2 with, $\alpha_j = -jc(M - N)$, $\beta_j = c$, $b_j = jM$, $j = 0, 1, \dots, m$.

The average unit purchase cost, thus, turns out to be, for $b_j \leq Q < b_{j+1}$, $j = 0, 1, \dots, m$,

$$\frac{C_j(Q)}{Q} = c - [jc(M - N)]\frac{1}{Q}, \quad b_j \leq Q < b_{j+1} \quad j = 0, 1, \dots, m. \quad (6)$$

Hence, the total cost function for the *carload discount schedule* becomes

$$TC_j(Q) = [k - jc(M - N)]\frac{d}{Q} + [dc - \frac{1}{2}jcc(M - N)] + \frac{ic}{2}Q. \quad (7)$$

A typical set of such curves is shown in Figure 2.

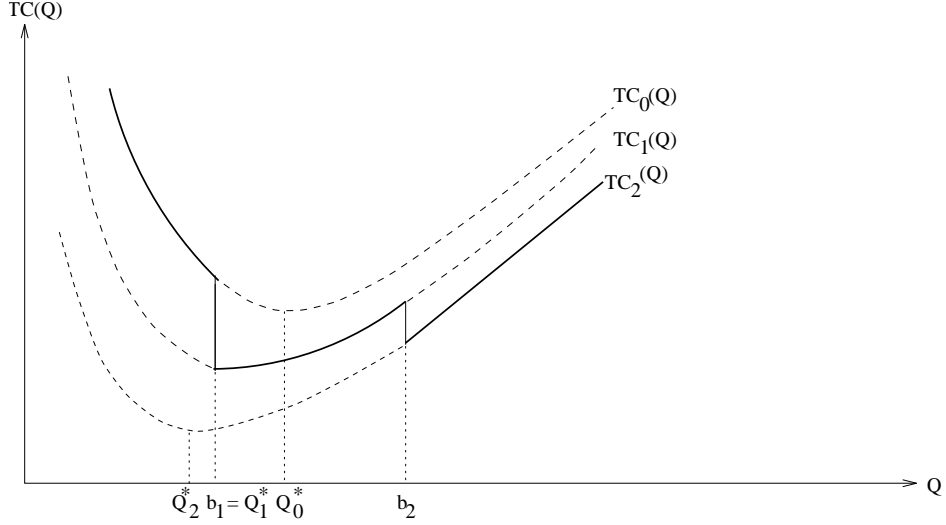


Figure 2: The total cost per unit time.

The total cost per unit time (7), is a convex function of Q if $k \geq jc(M - N)$ and a unique minimum can be found by differential calculus. When $k < jc(M - N)$, (7) becomes a nondecreasing concave function of Q , and minimum occurs at lowest value of Q , which is zero. For all nonnegative values of Q , the realizable cost function

$$TC(Q) = TC_j(Q), \quad b_j \leq Q < b_{j+1}, \quad j = 0, 1, \dots, m, \quad (8)$$

is shown in Figure 2 in bold lines.

For each curve, $TC_j(Q)$, the least total cost quantity, Q_j^* , is given by

$$Q_j^* = \sqrt{\frac{2d[k - jc(M - N)]}{ic}}, \quad j = 0, 1, \dots, m, \quad (9)$$

provided that $k - jc(M - N)$ is nonnegative. Furthermore, Q_j^* is said to be *realizable*, if it falls in the half interval, $b_j \leq Q_j^* < b_{j+1}$, $j = 0, 1, 2, \dots, m$.

3.2 Analysis of the Model

The total cost function per unit time, $TC(Q)$, is not a straightforward function to optimize. In general, it may take a form similar to the one depicted in Figure 3. In the following results, we show that the optimal solution will occur either at the minimum point of one of the convex total cost functions (provided that a realizable minimum exists) or at one of the price breakpoints to the right of it.

In order to devise a solution procedure for the optimal order quantity, the following results are needed. Proposition 1 shows that none of the cost curves intersects with one another, thus pointing to the possibility that optimum occurring at a price breakpoint. Then Proposition 3 shows that there can only be one realizable minimum. Proposition 4 provides a means to determine a minimum cost price breakpoint. Finally Proposition 5 shows that minimum cost realizable order quantity cannot occur in between two candidates for minimum price breakpoints. Thus it is sufficient to compare the best of two price points with the realizable minimum point, if one exists.

Proposition 1

$$TC_0(Q) > TC_1(Q) > \dots > TC_j(Q) > TC_{j+1}(Q) > \dots > TC_m(Q).$$

The proof is obvious. The result follows intuitively from the fact that the price per unit decreases as more units are ordered, i.e. buying the second set of Q units is always cheaper than buying the first set. Furthermore, it implies that none of the cost curves intersects with one another. [$TC_j(Q) = TC_{j+1}(Q)$, $j = 0, 1, \dots, m$, only if $M = N$, but this is the case with no discounts].

Proposition 2

$$Q_0^* > Q_1^* > \dots > Q_j^* > Q_{j+1}^* > \dots > Q_m^*,$$

provided that Q_j^* is defined.

Proof: Note that,

$$Q_j^* = \sqrt{\frac{2d[k - jc(M - N)]}{ic}}, \quad (10)$$

and

$$Q_{j+1}^* = \sqrt{\frac{2d[k - jc(M - N)]}{ic} - \frac{2dc(M - N)}{ic}}. \quad (11)$$

Since $\frac{2dc(M-N)}{i} > 0$, the result follows for all j such that $k \geq jc(M - N)$. \diamond

Q_j^* 's are only defined if $k \geq jc(M - N)$. As j increases, it may reach a value j_{\max} for which $k < j_{\max}c(M - N)$ and then, obviously, for all $j \geq j_{\max}$, $k < jc(M - N)$. At such points, the total cost $TC_j(Q)$ becomes a nondecreasing concave function. This situation is depicted in Figure 3 for $j_{\max} = 2$. This implies that as the number of price ranges increases, it becomes more and more reasonable to order at price breakpoints. The following result shows that there is only one non-breakpoint that can achieve the lowest costs.

Proposition 3 *There can be at most one realizable Q_j^* .*

Proof: First, it will be shown that if there exists an index l such that Q_l^* is defined and realizable, then there cannot exist any other index $j \neq l$ such that Q_j^* is realizable. Suppose that Q_l^* is defined and realizable, then $b_l \leq Q_l^* < b_{l+1}$ and $l < j_{\max}$. Thus, for all $r = 1, \dots, (j_{\max} - l - 1)$, we have $l + r > l$ and by Proposition 2, $Q_{l+r}^* < Q_l^* < b_{l+1}$. But, since, $b_l = lM$ and

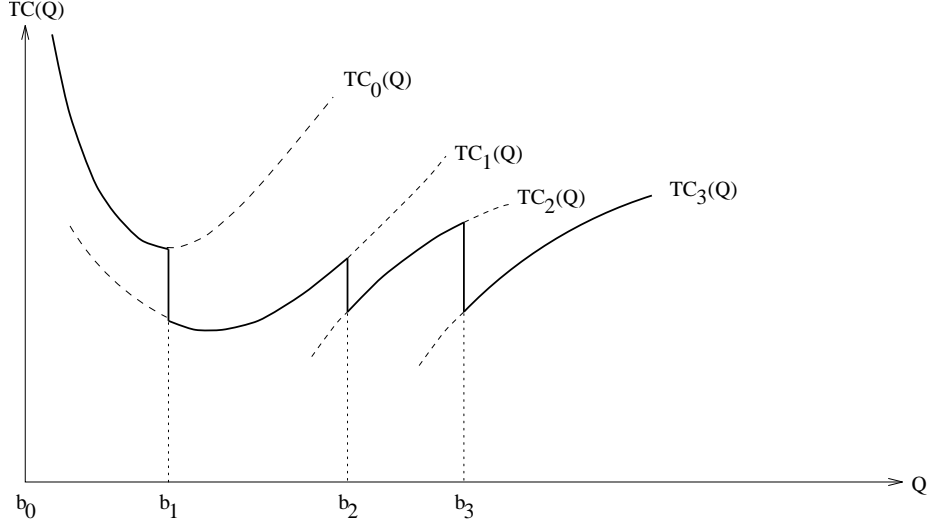


Figure 3: Appearance of concave cost functions.

$b_{l+1} < b_{l+r}$, we get $Q_{l+r}^* < b_{l+r}$, thus Q_{l+r}^* is not *realizable*. Similarly, for all $p = 1, \dots, l$, $Q_{l-p}^* > Q_l^*$ by Proposition 2. Using the fact that $Q_l^* > b_l$ and $b_l > b_{l-p+1}$ results in $Q_{l-p}^* > b_{l-p+1}$, which implies that Q_{l-p}^* is not *realizable*.

Hence, we have shown that, if an index l exists such that Q_l^* is defined and *realizable*, then there cannot be any other *realizable* Q_j^* for $j \neq l$. Next it will be shown that, even a single index l , with a *realizable* Q_l^* , may not exist.

Suppose that Q_l^* is *realizable*, that is $b_l \leq Q_l^* < b_{l+1}$. This implies that

$$lM \leq \sqrt{\frac{2d[k - lc(M - N)]}{ic}} < (l + 1)M. \quad (12)$$

The left inequality in (12) can be written as

$$l^2 + \left[\frac{2d(M - N)}{iM^2} \right] l - \frac{2dk}{icM^2} \leq 0. \quad (13)$$

Solving this quadratic inequality (taking the positive root), we get

$$l \leq \sqrt{\left[\frac{d(M-N)}{iM^2}\right]^2 + \frac{2dk}{icM^2}} - \frac{d(M-N)}{iM^2} \equiv l_{\max}. \quad (14)$$

Similarly, the right strict inequality in (12) can be written as

$$l^2 + \left[\frac{2d(M-N)}{iM^2} + 2\right]l - \frac{2dk}{icM^2} + 1 > 0. \quad (15)$$

And thus,

$$l > \sqrt{\left[\frac{d(M-N)}{iM^2}\right]^2 + \frac{2d(M-N)}{iM^2} + \frac{2dk}{icM^2}} - \frac{d(M-N)}{iM^2} - 1 \equiv l_{\min}. \quad (16)$$

Hence, it must be that $l_{\min} < l \leq l_{\max}$. If the half interval does not contain any integer, then no index l can exist such that Q_l^* is realizable. By the first part of the proof, we know that the half interval $(l_{\min}, l_{\max}]$ can contain at most one single integer. \diamond

Proposition 4 *To find the price breakpoint giving the minimum total cost per unit time, it is sufficient to compare the costs at two price breakpoints.*

Proof: For any price breakpoint, $b_j = jM$, $j = 0, 1, \dots, m$, the total cost per unit time is given by

$$TC_j(jM) = \frac{dk}{jM} + \frac{dcN}{M} + \frac{icN}{2}j \quad (17)$$

which is a convex function of j . Differentiating with respect to j and setting the resulting expression to zero, we get the optimal \hat{j} as $\sqrt{\frac{2dk}{icNM}}$. But this value for j will most likely be fractional. Using the convexity of the function, the index of the minimum cost breakpoint, j^* can be found by

$$TC_{j^*}(j^*M) = \min\{TC_{\lceil \hat{j} \rceil}(\lceil \hat{j} \rceil M), TC_{\lfloor \hat{j} \rfloor}(\lfloor \hat{j} \rfloor M)\}. \quad \diamond \quad (18)$$

Proposition 5 *There cannot be a realizable order quantity in the segment between the two candidate breakpoints for the minimal cost.*

Proof: Note that $\hat{j} = \sqrt{\frac{2kd}{icMN}} \geq \lfloor \hat{j} \rfloor$ and $Q_{\lfloor \hat{j} \rfloor}^* = \sqrt{\frac{2d[k - \lfloor \hat{j} \rfloor c(M-N)]}{ic}}$ is defined only if $k > \lfloor \hat{j} \rfloor c(M-N)$. Thus,

$$\frac{Q_{\lfloor \hat{j} \rfloor}^*}{\lfloor \hat{j} \rfloor M} = \sqrt{\frac{2d[k - \lfloor \hat{j} \rfloor c(M-N)]}{ic \lfloor \hat{j} \rfloor^2 M^2}} \quad (19)$$

and

$$\hat{j} = \sqrt{\frac{2kd}{icMN}} \geq \lfloor \hat{j} \rfloor, \quad (20)$$

so

$$\frac{Q_{\lfloor \hat{j} \rfloor}^*}{\lfloor \hat{j} \rfloor M} \leq \sqrt{\frac{N}{M} - \frac{\lfloor \hat{j} \rfloor cN(M-N)}{kM}} \leq \sqrt{\frac{N}{M}} < 1, \quad (21)$$

since $N < M$ by definition. Therefore, $Q_{\lfloor \hat{j} \rfloor}^* < \lfloor \hat{j} \rfloor M$ which implies that $Q_{\lfloor \hat{j} \rfloor}^*$ can never be realizable. \diamond

3.3 The Solution Procedure

Using the above results, the optimal lot sizing procedure is presented in figure 4. The solution procedure amount to the comparison of at most three candidate points for optimality. First l_{\max} and l_{\min} , the end points of the interval that contains indices for the realizable order quantities, are determined. It is shown by Proposition 3 that such an interval can contain at most one integer index. If such an index is found, then a single point, Q_l^* is the first candidate point, otherwise no realizable order quantity can be the optimal order quantity. Next, Proposition 4 is used to determine the indices of the price breakpoints giving the minimal total cost per unit time, namely

1. Compute

$$1.1. \quad l_{\max} = \sqrt{\frac{d^2(M-N)^2}{i^2M^4} + \frac{2dk}{icM^2}} - \frac{d(M-N)}{iM^2}$$

$$1.2. \quad l_{\min} = \sqrt{\left[\frac{d(M-N)}{iM^2}\right]^2 + \frac{2d(M-N)}{iM^2} + \frac{2dk}{icM^2}} - \frac{d(M-N)}{iM^2} - 1$$

$$1.2. \quad j = \sqrt{\frac{2kd}{icMN}}$$

2. **if** $\lfloor j \rfloor < m$, **then**

$$2.1. \quad j = \arg \min\{TC_{\lceil j \rceil}(\lceil j \rceil M), TC_{\lfloor j \rfloor}(\lfloor j \rfloor M)\}$$

$$2.2. \quad Q^* = jM$$

3. **else** $Q^* = jM$

4. **if** \exists an integer $l \in (l_{\min}, l_{\max}]$ **then**

$$4.1. \quad Q^* = \arg \min\{TC_j(jM), TC_l(Q_l^*)\}$$

Figure 4: Finding the optimal lot size under the carload discount schedule

$\lfloor j \rfloor$ and $\lceil j \rceil$. $\lfloor j \rfloor M$ and $\lceil j \rceil M$ are the two other candidate points for optimality. In the comparison of the candidate points based on their total costs per unit time, first $\lfloor j \rfloor M$ and $\lceil j \rceil M$ are compared, and the one with the minimal total cost per unit time is compared to the realizable order quantity Q_l^* , if any is found.

3.3.1 Numerical Example

Suppose that:

$$d = 2500 \text{ units per year,}$$

$$k = \$520 \text{ per order,}$$

$$c = \$25 \text{ per unit,}$$

$$M = 260 \text{ units,}$$

$$N = 250 \text{ units,}$$

$$m = 5 \text{ trucks,}$$

$$i = 20\% \text{ per year,}$$

then,

$$l_{max} \leftarrow \sqrt{\frac{d^2(M-N)^2}{i^2M^4} + \frac{2dk}{icM^2}} - \frac{d(M-N)}{iM^2} = 1.48.$$

$$l_{min} \leftarrow \sqrt{\left[\frac{d(M-N)}{iM^2}\right]^2 + \frac{2d(M-N)}{iM^2} + \frac{2dk}{icM^2}} - \frac{d(M-N)}{iM^2} - 1 = 0.99.$$

$$j^* \leftarrow \sqrt{\frac{2kd}{icMN}} = 2.82.$$

$$\lfloor j \rfloor = 2 < 5 = m;$$

$$TC_{\lfloor j \rfloor}(\lfloor j \rfloor M) = TC_{\lfloor j \rfloor}(b_{\lfloor j \rfloor}) = \frac{dk}{M_{\lfloor j \rfloor}} + \frac{dcN}{M} + \frac{icN_{\lfloor j \rfloor}}{2} = TC_2(b_2) = \$63,846.15.$$

$$TC_{\lceil j \rceil}(\lceil j \rceil M) = TC_{\lceil j \rceil}(b_{\lceil j \rceil}) = \frac{dk}{M_{\lceil j \rceil}} + \frac{dcN}{M} + \frac{icN_{\lceil j \rceil}}{2} = TC_3(b_3) = \$63,637.82.$$

Since $TC_3(b_3) < TC_2(b_2)$ **then** $j \leftarrow 3$

Since $l = 1 \in (0.99, 1.48]$ **then** Q_1^* is realizable.

$$Q_1^* = \sqrt{\frac{2d[k-c(M-N)]}{ic}} = 519.6 \text{ units.}$$

$$TC_1(Q_1^*) = [k - c(M - N)] \times \frac{d}{Q_1^*} + dc - \frac{i}{2}c(M - N) + \frac{ic}{2}Q_1^* = 67,205.84$$

Since $TC_j(jM) = TC(3 \times 260) < TC_1(Q_1^*) = TC(519.6)$ then $Q^* = 780$ units.

4 Conclusions

Harris' EOQ model, in spite of the strict assumptions in its formulation, is still one of the most widely used models in inventory management systems. The use of this model in integrated inventory management systems makes it necessary to incorporate quantity discounts in the cost structure. This is especially important when transportation costs are incorporated in the analysis. The carload discount model seems to be a simple way of doing this.

Contrary to the expectations reported in the literature (e.g. (Nahmias 2001)), optimal lot sizing for the *carload discounts* schedule turns out to be extremely simple. The solution procedure is basically the comparison of at most three candidate points for optimality.

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References

- W. C. Benton and S. Park. A classification of literature on determining the lot size under quantity discounts. *European J. Operational Research*, 92: 219–238, 1996.
- C. W. Churchman, R. L. Ackoff, and E. L. Arnoff. *Introduction to Operations Research*. Wiley, 1957.
- R. J. Dolan. Quantity discounts: Managerial issues and research opportunities. *Marketing Science*, 6:1–22, 1987.
- G. Hadley and T. M. Whitin. *Analysis of Inventory Systems*. Prentice-Hall, 1963.
- F. W. Harris. How many parts to make at once. *Factory, The Magazine of Management*, 10:135–136,152, 1913. Reprinted in *Ops Res.*, 38:947–950, 1990.
- S. Nahmias. *Production and Operations Analysis, 4th ed.* McGraw-Hill/Irwin, 2001.
- T. M. Whitin. *The Theory of Inventory Management*. Princeton University Press, 1953.