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Theory and Methodology

Optimal streaming of a single job in a two-stage flow shop

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Abstract

Lot streaming is moving some portion of a process batch ahead to begin a downstream operation. The problem to be considered in this paper is the following: a single job consisting of U units is to be processed on two machines in the given order. Given a fixed number of possible transfer batches between the two machines, the problem is to find the timing and the size of the transfer batches (or, sublots) so as to optimize a given criterion. The schedules can be evaluated based on job completion, sublot completion, or item completion times. In the single job lot streaming problem, minimizing job completion time corresponds to minimizing the makespan, for which formulas for optimal sublot sizes are available. In this paper, the results for the sublot and item completion time models are presented.

Keywords: Production; Scheduling theory; Lot streaming; Flow shops

1. Introduction

Lot streaming is moving some portion of a process batch ahead to begin a downstream operation allowing for the downstream operation to start earlier and hence improving the given criterion. Since the initial papers of Baker [1] and Trietsch [2], there has been considerable interest in the area. For a review of the literature, the reader is referred to [3,4]. In addition to its implications in Group Technology (leading to cell based manufacturing, resulting in shorter lead times and reduced work in progress inventories), Just-in-Time Systems (*lot size of one*) and OPT/Synchronous Manufacturing (*transfer vs. process batches*), future significance of lot streaming techniques may possibly lie in the integration of production planning and machine scheduling. One problem with production planning models is that the sequencing requirements on the resources cannot be easily accommodated, whereas machine scheduling models cannot explicitly handle lot sizing.

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In lot streaming it is assumed that a number of *jobs* each consisting of a number of identical parts (or, items) are to be processed on a sequence of machines. In traditional machine scheduling models the items are available to be moved to the next machine only when *all* items in the job complete processing on the current machine. If it is feasible to partition the job into sublots and transport each sublot to the next operation independent of other sublots, then the rate at which the parts reach the last machine increases, resulting in improved completion time of the parts in the last machine. As pointed out by Potts and Van Wassenhove [5].

> Most scheduling models assume that no shipment is possible until the entire job is completed. However, in this case, the customer may be out of stock while awaiting delivery. Assume, for example, that a customer has a low inventory of some product and places a replenishment order consisting of a number of pallets to cover expected demand for the next few months. It may take several weeks to process the complete order. However, the customer service is improved, firstly by producing a few pallets in the near future to cover the customer's demand during the month, and then by satisfying the remaining part of the order at some later date. It is now apparent that if items or sublots may be shipped immediately upon completion, decomposing a job into sublots may improve customer service.

An important feature of modeling lot streaming problems is the time when the parts in the last machine become *available to be shipped* to an (external or internal) customer. In their review of batching and lot-sizing literature, Potts and van Wassenhove [5] proposed three models:

Job completion time model. The shipment to the customer can take place only when all the parts in the job complete processing in the last machine.

Sublot completion time model. Each sublot can be shipped to the customer independently. The shipment to the customer takes place when all the parts in sublot complete processing in the last machine.

Item completion time model. Each part can be shipped to the customer whenever it completes processing in the last machine.

For each scheduling problem, the technology of the shop dictates whether or not lot streaming is allowed; and, if it is allowed, which of the above three type of models is appropriate for the problem. In all of the above models the sublots are available to be moved to the next machine, as soon as they are completed in the current machine. The models differ primarily as to when the items in the last machine become available to be moved to the "customer". In some cases, although the order is transported in sublots within the factory, it is delivered to the customer as a whole. This situation is depicted in the job completion time model. In other cases, the order may be shipped to the customer in batches (sublots), as in the sublot completion time models. Customer service may further be improved if each unit is shipped to the customer individually as soon as its processing is completed in the last machine. This situation can be represented by an item completion time model. All other things being equal, the optimal solution to the item completion time model gives the shortest average time that a unit stays in the work-in-process inventory and the maximum customer service. Thus, lot streaming, in addition to improving customer service, reduces the average work-in-process inventory in the shop.

This paper deals with optimal streaming a single job in two machine (flow) shop. Although there may be actual problems with these exact structures in which optimal solutions are desirable, the main motivation for this study was to investigate the analytical and computational problems in designing optimizing algorithms for the simplest non-trivial case. The results obtained can be used as building blocks and provide insights in designing algorithms for more complex shop structures, and provide bounds on the heuristic approaches.

The next section defines the problem and presents a review of previous work in this area. In the following section, results are presented for sublot completion time model. Details of the proofs are given in Appendix A. The final section summarizes the insights gained in this analysis.

2. The problem

The problem to be considered in this paper is the following: a single job consisting of U units is to be processed on two machines. Let p_i, $i = 1, 2$, be the processing time of each unit on machine i. There are s sublots on each machine. In this study, we shall assume that the sublots are *consistent*. That is, the sublots are same size at each machine. L_k, $k = 1, \dots, s$ is the size of (i.e. the number of units in) the kth sublot, $\sum_{k=1}^{s} L_k = U$. Furthermore, the analysis in this paper will be based on allowing for fractional number of units in a sublot. If the job contains large number of units, then this is an acceptable assumption. On the other hand, if the number of units in the job is such that the fractional sublot sizes are not meaningful, then one has to impose the integrality requirement on the sublot sizes ($L_k \geqslant 0$, and *integer*), resulting in a mixed-integer program. C_{ik} is the completion time of kth sublot on machine i. The problem is to find the timing and the size of the sublots on each stage so as to minimize a given optimality criterion.

Under these assumptions, the problem can be described by the following constraints.

$$C_{ik} \geqslant C_{i,k-1} + p_i L_k, \quad i = 1, 2, \quad k = 1, \dots, s, \tag{1}$$

$$C_{2k} \geqslant C_{1k} + p_2 L_k, \qquad k = 1, \dots, s, \tag{2}$$

$$\sum_{k=1}^{s} L_k = U, \tag{3}$$

$$C_{i0} = 0, \qquad i = 1, 2, \tag{4}$$

$$C_{ik} \geqslant 0, \qquad i = 1, 2, \qquad k = 1, \dots, s, \tag{5}$$

$$L_k \geqslant 0, \qquad k = 1, \dots, s. \tag{6}$$

Constraints (1) allow sublot k to start only after sublot $k-1$ is completed. Constraints (2) prevent a sublot to be processed simultaneously on both machines. Constraint (3) assures the sublots to account for all the units in the job.

Various performance criteria can be used with these constraints. For example, in the job completion time model, defined earlier, the objective function is minimizing C_{2s} subject to Constraints (1)–(6). Note that there is no distinction between makespan and total flowtime when there is one job in the problem. In other words, minimizing total flowtime corresponds to minimizing makespan, or C_{2s}. The *m-machine* version of this problem was first studied, independently, by Baker [1] and Trietsch [2]. Baker [1] gave the linear programming formulation of the problem. Baker [1] and Potts and Baker [6] showed that for two-*machine* case, the closed form solution is given by the geometric sublot sizes

$$L_1 = U \frac{1-\pi}{1-\pi^s}, \tag{7}$$

$$L_k = \pi L_{k-1}, \quad k = 2, \dots, s, \tag{8}$$

where $\pi \equiv p_2/p_1$. If one uses equal sublot sizes, which are more convenient in practice, Potts and Baker [6] have shown that, $F^{\mathrm{E}}(L)/F^*(L) < 1.09$, where $F^*(L)$ is the optimal job completion time when geometric sublot sizes are used and $F^{\mathrm{E}}(L)$ is the job completion time when equal sublot sizes are used.

In sublot completion time model an item leaves the shop when the sublot to which it belongs is completed in the second machine. Total flowtime of all units in the job is $\sum_{k=1}^{s} L_k C_{2k}$. Minimizing total flowtime is equivalent to minimizing the average time a unit spends in the shop, i.e. the mean flowtime, $(1/U)\sum_{k=1}^{s} L_k C_{2k}$. Note that, this is not "weighted" mean flowtime in the traditional sense, since L_k's cannot be exogenously assigned, but are endogenous variables of the problem. The problem becomes a quadratic programming problem with the objective function minimizing $\sum_{k=1}^{s} L_k C_{2k}$ subject to Constraints (1)–(6). This model is analyzed further in Section 3.

Finally, in the item completion time model an item is assumed to be completed as soon as it completes processing in the last machine. When continuous sublot sizes are allowed, this is equivalent to assuming "infinite" number of transfers in the last machine (in other words, after the first sublot starts in the second machine, parts in the second machine become available to be shipped to a customer "continuously"). In the case of two-machine flow shop with consistent sublot sizes, the objective function is $\min \sum_{k=1}^{s} [C_{2k} - (p_2/2)L_k]L_k$ subject to Constraints (1)–(6). There are two cases to consider: (i) $\pi \leqslant 1$, and (ii) $\pi > 1$ where $\pi \equiv p_2/p_1$. Çetinkaya and Gupta [7] have shown if $\pi \leqslant 1$, then equal size sublots are optimal, otherwise one has to use the geometric sublot sizes given in Eqs. (7) and (8). Again, as in the case of job completion model, it may be more convenient to use equal size sublots in practice, even when $\pi > 1$. For notational convenience, let $U = 1$, $p_1 = 1$, and $p_2 = \pi$, and denote by $F^*(L)$ the optimal mean item completion time when geometric sublot sizes are used, and $F^{\mathrm{E}}(L)$ be the mean item completion time when equal sublot sizes are used. It can be observed in Figs. 1 and 2, that $F^{\mathrm{E}}(L) = 1/s + \frac{1}{2}\pi$, and $F^*(L) = (\pi - 1)/(\pi^s - 1) + \frac{1}{2}\pi$. It can be shown, using similar computations as in Section 3.2, that $F^{\mathrm{E}}(L)/F^*(L) < 1.18$. (Numerical solutions indicate the largest value for the ratio to be 1.172 at $s = 4$ and $\pi = 2.021$.)

3. The sublot completion time model

The model to be discussed in this section is the sublot completion model of streaming a single job in a two-machine shop. In this model, each sublot can be shipped to the customer independently. The shipment to the customer takes place when all the parts in sublot complete processing in the second machine. The problem is a quadratic programming problem with the objective function

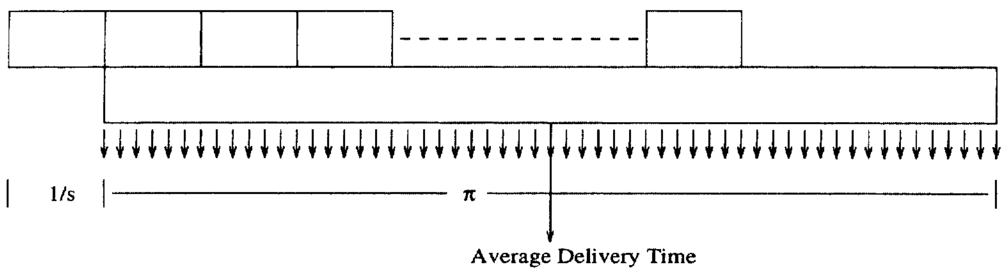

Fig. 1. Item completion, equal sublot sizes.

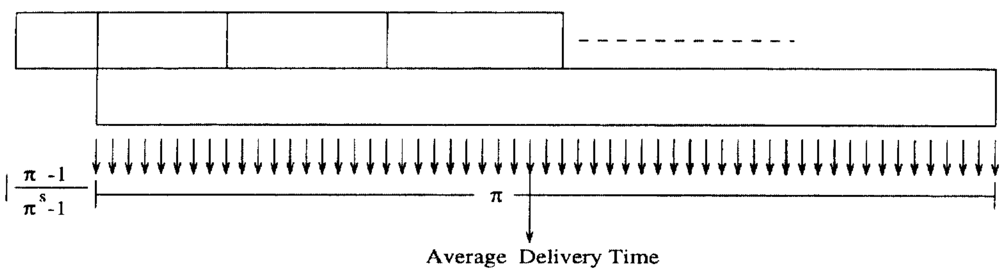

Fig. 2. Item completion, optimal sublot sizes.

$$\sum_{k=1}^{s} L_k C_{2k} \tag{9}$$

subject to Constraints (1)–(6).

This quadratic objective function was first proposed by Kropp and Smunt [8]. When the sublots are assumed to be consistent, they gave a complete formulation for m-machine problem using the linear constraints proposed by Baker in [1]. When there are only two sublots allowed between the machines, i.e., $s = 2$, efficient algorithms are proposed by Çetinkaya and Gupta [7] and Topaloğlu et al. [9].

For the two-machine problem, letting $\pi \equiv p_2/p_1$, we have the following two results:

Result 1. *If $\pi \leqslant 1$, then the sublots of equal size are optimal, i.e.*

$$L_k^* = U/s, \quad k = 1, \ldots, s. \tag{10}$$

Result 2. *If $\pi > 1$ optimal sublots can be found by the following algorithm.*

Algorithm 1

$v \leftarrow 0$, optimal $\leftarrow$ FALSE
If $f(\pi) = -\pi^{2s} + 2\pi^{s+1} + 2\pi^{s} - 2\pi - 1 > 0$
  optimal $\leftarrow$ TRUE, geometric sublots are optimal
While not optimal

  $v \leftarrow v + 1$
  $\bar{L}_1 \leftarrow U[\frac{\pi^{v}-1}{\pi-1}\pi - (s-v)]/[\frac{\pi^{2v}-1}{\pi^2-1}\pi(s-v) + (\frac{\pi^{v}-1}{\pi-1})^2\pi]$
  $\bar{L}_k \leftarrow \pi^{k-1}\bar{L}_1, \quad k = 2, \ldots, v$
  $\bar{L}_k \leftarrow [U - \bar{L}_1 \sum_{l=1}^{v} \pi^{l-1}]/[(s-v)], \quad k = v+1, \ldots, s$
  if $\pi \bar{L}_v \geqslant \bar{L}_{v+1} \geqslant \bar{L}_v$,
    optimal $\leftarrow$ TRUE, $\bar{L} = \bar{L}_1, \ldots, \bar{L}_s)$ is optimal

Basically, the above algorithm creates geometric sublot sizes (with ratio π) up to sublot v, followed by equal sublot sizes for the rest of the schedule. The algorithm then checks a necessary condition to see if the current value of v is optimal; if not, it looks for a better choice of v. The search starts at $v = 1$ and stops as soon as the necessary condition is satisfied.

Result 1 is due to Çetinkaya and Gupta [7]; an alternate proof is given in Section A.1. Result 2 was conjectured but not proven in Çetinkaya and Gupta [7]; a detailed proof of this result is given in Section A.2.

It is difficult to give an intuitive explanation for the resulting sublot sizes obtained by the above algorithm. It is not possible to have a general closed form solution for the sublot sizes. For certain values of s, though, closed form solutions can be obtained. For example, for $s = 2$ and $s = 3$, optimal sublot sizes are given below.

Solution for $s = 2$

$$(L_1^*, L_2^*) = \begin{cases} \left(\frac{1}{\pi+1}, \frac{\pi}{\pi+1}\right) & \text{if } 1 \leqslant \pi \leqslant 1 + \sqrt{2}\,, \\ \left(\frac{\pi-1}{2\pi}, \frac{\pi+1}{2\pi}\right) & \text{if } 1 + \sqrt{2} \leqslant \pi\,. \end{cases} \tag{11}$$

Solution for $s = 3$

$$(L_1^*, L_2^*, L_3^*) = \begin{cases} \left(\frac{1}{\pi^2+\pi+1}, \frac{\pi}{\pi^2+\pi+1}, \frac{\pi^2}{\pi^2+\pi+1}\right) & \text{if } 1 \leqslant \pi \leqslant (1+\sqrt{5})/2\,, \\ \left(\frac{1}{2}\frac{\pi^2+\pi-1}{\pi^3+\pi^2+\pi}, \frac{1}{2}\frac{\pi^3+\pi^2-\pi}{\pi^3+\pi^2+\pi}, \frac{1}{2}\frac{\pi^3+2\pi+1}{\pi^3+\pi^2+\pi}\right) & \text{if } (1+\sqrt{5})/2 \leqslant \pi \leqslant (3+\sqrt{13})/2\,, \\ \left(\frac{\pi-2}{3\pi}, \frac{\pi+1}{3\pi}, \frac{\pi+1}{3\pi}\right) & \text{if } (3+\sqrt{13})/2 \leqslant \pi\,. \end{cases} \tag{12}$$

3.1. Variable size sublots

The above results for sublot completion time model are based on the requirement that sublots are *consistent*, i.e. the size of the kth sublot is L_k in both of the machines. Suppose we relax this requirement, and thus define *variable* size sublots as $L_{ik},\ i = 1,\dots,s$ to be the number of units in the kth sublot on machine i. One would expect an improvement in the objective function value when sublot sizes are allowed to vary. Feasible solutions with consistent sublot sizes constitute a proper subset of the set of feasible solutions with variable size sublots; therefore, the optimal solution with variable size sublots must be at least as good as the optimal solution with consistent sublot sizes. In the two-machine job completion time model, as pointed out by Trietsch and Baker [3], this relaxation will not improve on the optimal solution with consistent sublot sizes, since there is only one set of transfers between the machines – from the first machine to the second. In job completion time model, there is only one transfer from the second machine: items become available to be shipped to the customer only when last sublot is completed in the second machine.

This is not necessarily the case in the two-machine sublot completion time models, as the following example illustrates. Suppose that a job consisting of 60 units is to be processed in a two-machine flow shop using at most two sublots. Let the processing times be $p_1 = 1$ and $p_2 = 3$. With $s = 2$ and $\pi = 3$, from expression (11), $(L_1^*, L_2^*) = (20, 40)$; resulting in a mean sublot completion time (a mean flowtime) of $\frac{1}{60}(20 \times 80 + 40 \times 200) = 160$. But a better mean sublot completion time of $\frac{1}{60}(30 \times 105 + 30 \times 195) = 150$ can be achieved using $(L_{11}^*, L_{12}^*) = (15, 45)$ and $(L_{21}^*, L_{22}^*) = (30, 30)$ (see Fig. 3).

This leads us to consider the following conjectures:

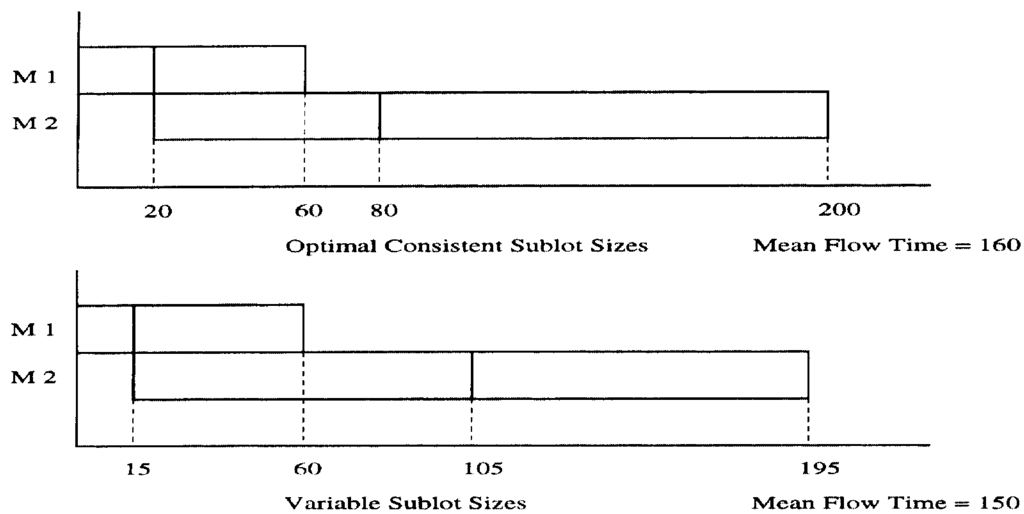

Fig. 3. Sublot completion, non-optimality of consistent sublots.

Conjecture 1. Mean sublot completion time is minimized by equal sublots on each stage if $p_1 \geqslant p_2$.

Conjecture 2. Mean sublot completion time is minimized by geometric sublots on first stage, and equal sublots on second stage if $p_1 < p_2$.

The following intuitive justification can be provided for these conjectures: *processing should be continuous on the dominant machine and start as early as possible.* For $p_1 \geqslant p_2$, the first machine is dominant and determines the sublot sizes. For $p_1 < p_2$, the dominant machine is the second one and geometric sublots on the first machine allow the second machine to start continuous production as early as possible.

3.2. Equal size sublots

If $p_1 \geqslant p_2$, then equal size sublots are optimal for the problem with consistent sublot sizes. (If Conjecture 1 is true, then equal size sublots are optimal even when *variable* size sublots are allowed.) As with the other models, for practical reasons, one may choose to use equal size sublots also in the case when $p_1 < p_2$. It will be useful to determine how much impairment this approximation will cause in the mean sublot completion time. For notational convenience, let $U = 1$, $p_1 = 1$, and $p_2 = \pi$; when $p_1 < p_2$, i.e. $\pi > 1$, mean sublot completion time with equal sublot sizes (see Fig. 4) is

$$\begin{aligned} F^{\mathrm{E}}(L) &= \left(\frac{1}{s} + \pi\frac{1}{s}\right)\frac{1}{s} + \left(\frac{1}{s} + \pi\frac{2}{s}\right)\frac{1}{s} + \cdots + \left(\frac{1}{s} + \pi\frac{s}{s}\right)\frac{1}{s} \\ &= \frac{1}{s} + \pi\frac{1}{s^2}\sum_{k=1}^{s} k \\ &= \frac{1}{s} + \pi\frac{(s+1)}{2s}. \end{aligned} \tag{13}$$

Since it is not possible to derive explicit expression for the optimal mean sublot completion time using consistent sublots, $F^{\mathrm{C}}(L)$, we shall use a lower bound for its value. We know (from Result 7 in Appendix A) that,

$$\pi L_k \geqslant L_{k+1}, \quad k = 1, \ldots, s-1$$

is a necessary condition for optimality.

Consider the following linear program:

$$z = \min L_1$$

$$\text{subject to} \quad \pi L_k \geqslant L_{k+1}, \quad k = 1, \ldots, s-1,$$

$$\sum_{k=1}^{s} L_k = 1, \qquad L_k \geqslant 0, \quad k = 1, \ldots, s-1.$$

It is not difficult to show that $z = (\pi - 1)/(\pi^s - 1)$. Thus, the smallest possible size of the first sublot on the first machine is z. Since $p_1 = 1$, z is the earliest time the second machine can start processing. Once the second machine starts processing, it will continue uninterrupted because $\pi > 1$ and the first constraint, $\pi L_k \geqslant L_{k+1}$, $k = 1, \ldots, s-1$. Thus a lower bound for the optimal sublot completion time, $F^{\mathrm{C}}(L)$, is given by the minimal value of the following quadratic program:

$$F^{\mathrm{LB}}(L) = \{\min[z + \pi L_1]L_1 + [z + \pi(L_1 + L_2)]L_2 + \cdots + [z + \pi(L_1 + L_2 + \cdots + L_s)]L_s\}$$

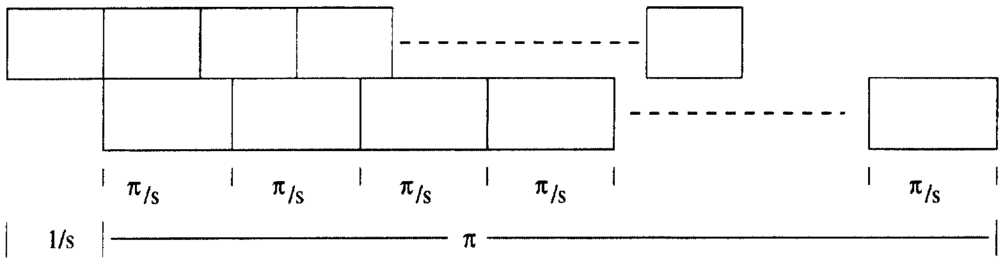

Fig. 5. Sublot completion, lower bound on consistent sublot sizes.

$$\text{subject to} \quad \sum_{k=1}^{s} L_k = 1, \qquad L_k \geqslant 0, \quad k = 1, \ldots, s,$$

which has the solution (see Fig. 5), $L_k = 1/s$ and

$$F^{\mathrm{LB}}(L) = \frac{(\pi - 1)}{(\pi^s - 1)} + \pi\frac{(s+1)}{2s}. \tag{14}$$

From Expressions (13) and (14), we have

$$F^{\mathrm{E}}(L)/F^{\mathrm{LB}}(L) = \frac{(1/s) + \pi(s+1)/2s}{(\pi - 1)/(\pi^s - 1) + \pi(s+1)/2s}.$$

Suppose s can take any real value, then $f(\pi, s) = F^{\mathrm{E}}(L)/F^{\mathrm{LB}}(L)$ is a continuous function of s and π for $s \geqslant 2$ and $\pi > 1$. Setting the partial derivatives of $f(\pi, s)$ with respect to s and π equal to zero, and solving the resulting two non-linear equations numerically, gives a unique solution $(\pi^*, s^*) = (1.938, 4.267)$. (Since $[\partial^2 f(\pi^*, s^*)/\partial\pi\partial s]^2 - (\partial^2 f(\pi^*, s^*)/\partial\pi^2)(\partial^2 f(\pi^*, s^*)/\partial s^2) = -0.00184$, (π^*, s^*) is a maximum point.)

Clearly, for discrete values of s, the maximum of $f(\pi, s)$ should be less than $f(\pi^*, s^*) = f(1.938, 4.267) = 1.14$. For example, $f(\hat{\pi}, \hat{s}) = f(1.992, 4) = 1.139$. Thus, we have the following result.

Result 3. $F^{\mathrm{E}}(L)/F^{\mathrm{LB}}(L) < 1.14$.

Since $F^{\mathrm{LB}}(L) \leqslant F^{\mathrm{C}}(L)$, $F^{\mathrm{E}}(L)/F^{\mathrm{C}}(L) \leqslant F^{\mathrm{E}}(L)/F^{\mathrm{LB}}(L)$, thus by Result 3, $F^{\mathrm{E}}(L)/F^{\mathrm{C}}(L) < 1.14$.

It is interesting to observe that the sublot sizes found while constructing a lower bound for the model with consistent sublot sizes are "geometric sublot sizes in the first machine" and "equal size sublot sizes

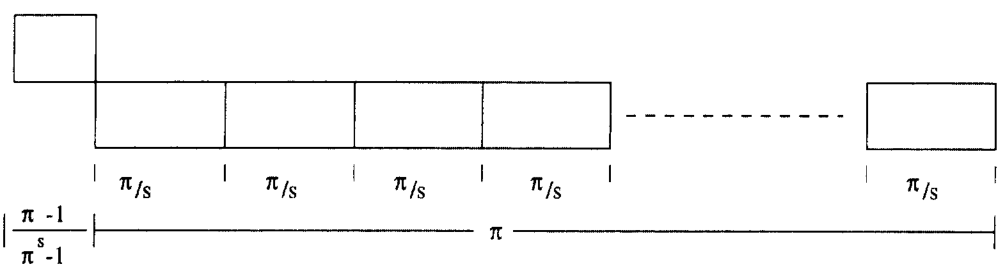

Fig. 4. Sublot completion, equal sublots.

in the second machine"; which is the claim in Conjecture 2. If this conjecture is true, then $F^{E}(L)/F^{*}(L) < 1.14$, where $F^{*}(L)$ is the mean sublot completion time achievable by allowing for variable size sublots.

4. Conclusions

In this paper we have analyzed lot streaming of a single job in a two-stage flow shop. Where applicable, the distinction between the consistent and variable size sublots were emphasized. The implications of equal sublots, which are widely used in practice were also presented. Table 1 summarizes the results. (In the item completion time model, the sublot sizes to be determined are between the first and the second machines, since the items leave the second machine one at a time as they are completed; therefore, the distinction between consistent and variable size sublots in this model is not relevant.)

Even when variable sublot sizes are allowed, consistent sublots are optimal in all cases, except in sublot completion time model with $p_1 < p_2$. As seen from the last column of Table 1, equal sublots are quite effective, justifying the use of equal sublots in practice.

Acknowledgement

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Appendix A

In this appendix detailed proofs of the results obtained in sublot completion time model are presented. Result 1 is due to Çetinkaya and Gupta [7]. An alternate proof is given below. Then it is shown that the algorithm in Result 2 finds the optimal solution.

A.1. Result 1

If $p_2/p_1 \equiv \pi \leqslant 1$, then the sublots of equal size are optimal for the sublot completion time, i.e.

$$L_k^* = U/s, \quad k = 1, \ldots, s.$$

Table 1
Sublot sizes

		Consistent	Variable	Equal/optimal
Job completion	$p_1 \geqslant p_2$	Geometric	Geometric	1.09
	$p_1 < p_2$	Geometric	Geometric	1.09
Sublot completion	$p_1 \geqslant p_2$	Equal	Equal [a]	1.00 [a]
	$p_1 < p_2$	Algorithm 1	Machine 1: Geometric [a] Machine 2: Equal [a]	1.14 [a]
Item completion	$p_1 \geqslant p_2$	Equal	Equal	1.00
	$p_1 < p_2$	Geometric	Geometric	1.18

[a] Conjectured.

Proof. Consider the general case: there are m machines, with the property $p_1 \geqslant \max_{2 \leqslant i \leqslant m}\{p_i\}$, and we will show that equal sublot sizes (i.e. $L_k = U/s$, $k = 1, \dots, s$) are optimal. Without loss of generality, assume that $U = 1$, then we have the following formulation for an m-machine flow shop.

$$\begin{aligned} \min \quad & F(L) = \sum\nolimits_{k=1}^{s} L_k C_{mk} \\ \text{subject to} \quad & \sum\nolimits_{k=1}^{s} L_k = 1 \ , \\ & C_{11} - p_1 L_1 \geqslant 0 \ , \end{aligned}$$

$$\begin{aligned} C_{ik} - C_{i,k-1} - p_i L_k \geqslant 0, \quad & i = 1, \dots, m, \quad k = 2, \dots, s, \\ C_{ik} - C_{i-1,k} - p_i L_k \geqslant 0, \quad & i = 1, \dots, m, \quad k = 2, \dots, s. \\ C_{ik} \geqslant 0, \quad & i = 1, \dots, m, \quad k = 2, \dots, s, \\ L_k \geqslant 0, \quad & k = 1, \dots, s. \end{aligned}$$

We first need the following result showing that there exists an optimal solution with non-decreasing sublot sizes.

Result 4. *If $p_1 \geqslant \max_{2 \leqslant i \leqslant m}\{p_i\}$ then an optimal solution exists in which,*

$$L_k \leqslant L_{k+1}, \quad k = 1, \dots, s. \tag{15}$$

Proof. Suppose the contrary, that is, there exists an optimal solution $L = (L_1, \dots, L_s)$ such that for at least one k, $L_k > L_{k+1}$.

Now we will give an algorithm that will construct a schedule satisfying Condition (15) and having the objective value not more than that of L. Let $\Pi_t = (\Pi_t(1), \Pi_t(2), \dots, \Pi_t(s))$ and $\overline{\Pi} = (\overline{\Pi}(1), \overline{\Pi}(2), \dots, \overline{\Pi}(s))$ denote the sublot sequence at tth iteration of the algorithm and the optimal sublot sequence, respectively.

```
Π_0 ← Π̄
for t = 0 to s − 1 do
  begin
    r ← arg max_{1 ≤ k ≤ s−t} {L_{Π_t(k)}}
    Π_{t+1} ← Π_t
    for k = r to s − t do
      Π_{t+1}(k) ← Π_t(k + 1)
    Π_{t+1}(s − t) ← Π_t(r)
  end
```

In the tth iteration of first **for** loop, the minimum sublot among the first $(s - t)$ sublots in the sequence Π_t is removed from its place and inserted in $(s - t)$th place to form the sequence Π_{t+1}. The final schedule satisfies

$$L_k \leqslant L_{k+1}.$$

To show that the resulting schedule has objective value not more than the optimal solution, consider the following two observations:

Result 5. *In the tth iteration of the algorithm, if the largest sublot among the first $(s - t)$ sublots, $\Pi_t(r)$, is removed from the schedule, then the minimum decrease in the total flowtime Δ^- is*

$$\Delta^- \geqslant p_1 L_{\Pi_t(r)} \sum_{l=r+1}^{s} L_{\Pi_t(l)}.$$

Proof. Let C^- be the minimum decrease in the completion time of the sublots that follow the removed sublot. Clearly,

$$\Delta^- \geqslant C^- \sum_{l=r+1}^{s} L_{\Pi_t(l)}.$$

As illustrated in Fig. 6, a lower bound on the C^- can be found as

$$C^- \geqslant \min_{\{1 \leqslant i \leqslant m\}} \{C_{i,\Pi_t(r)} - C_{i,\Pi_t(r-1)}\}.$$

The lot streaming problem turns out to be an ordered flow shop problem, when the sublot sizes are given. This result follows from two facts. First, if a sublot has the kth largest processing time on machine i, then it has the kth largest processing time on every other machine. Second, if a sublot has its ith largest processing time on machine ℓ, then every other sublot has the ith largest processing time on machine ℓ. In ordered flow shops, when the first machine has the largest processing time, the minimum makespan is achieved, if the jobs are in non-increasing order of processing times. This result is given in Smith et al. [10]. In order to prove this result, they consider an optimal sequence with at least one job whose processing time is larger than the processing time of the immediately succeeding job. They show that, the new sequence formed by pairwise interchange of these two jobs does not have a longer makespan. This implies, however, that the longest makespan is achieved when the jobs are in non-decreasing order of processing times. Using this fact, we get

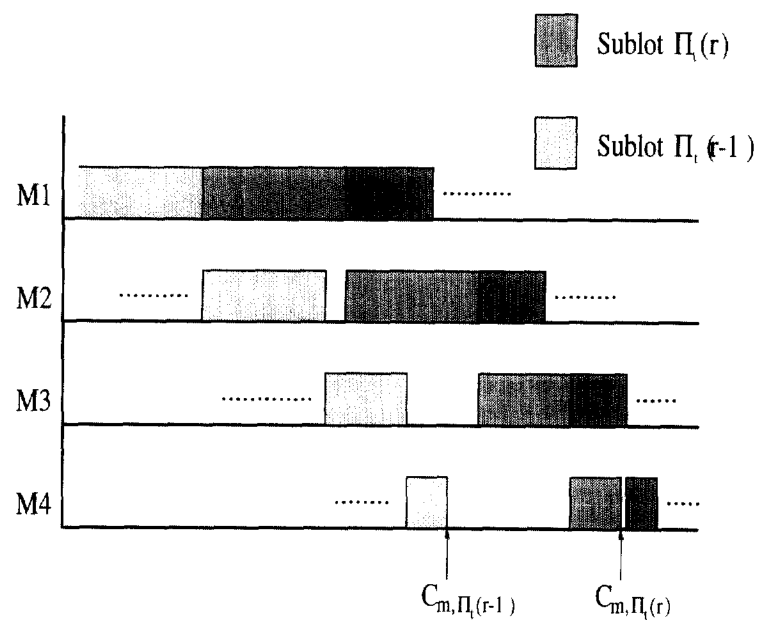

Fig. 6. Gantt chart of Π_t.

$$C_{i,\Pi_t(r-1)} \leqslant p_1 \sum_{l=1}^{r-1} L_{\Pi_t(l)} + \max_{\{1 \leqslant l \leqslant r-1\}} \{L_{\Pi_t(l)}\} \sum_{v=2}^{i} p_v.$$

The expression on the right is the maximum completion time that can be achieved on the ith machine by sequencing the first $r-1$ sublots. The following is a lower bound on $C_{i,\Pi_t(r)}$:

$$C_{i,\Pi_t(r)} \geqslant p_1 \sum_{l=1}^{r} L_{\Pi_t(l)} + L_{\Pi_t(r)} \sum_{v=2}^{i} p_v.$$

Thus, we have

$$\begin{aligned} C^- &\geqslant \min_{\{1 \leqslant i \leqslant m\}} \left\{ p_1 \sum_{l=1}^{r} L_{\Pi_t(l)} + L_{\Pi_t(r)} \sum_{v=2}^{i} p_v - \left(p_1 \sum_{l=1}^{r-1} L_{\Pi_t(l)} + \max_{\{1 \leqslant l \leqslant r-1\}} \{L_{\Pi_t(l)}\} \sum_{v=2}^{i} p_v \right) \right\}, \\ &\geqslant \min_{\{1 \leqslant i \leqslant m\}} \left\{ p_1 L_{\Pi_t(r)} + \left(L_{\Pi_t(r)} - \max_{\{1 \leqslant l \leqslant r-1\}} \{L_{\Pi_t(l)}\} \right) \sum_{v=2}^{i} p_v \right\}. \end{aligned}$$

Since $L_{\Pi_t(r)} = \max_{\{1 \leqslant l \leqslant r\}} \{L_{\Pi_t(l)}\} \geqslant \max_{\{1 \leqslant l \leqslant r-1\}} \{L_{\Pi_t(l)}\}$ we have

$$\begin{aligned} C^- &\geqslant \min_{\{1 \leqslant i \leqslant m\}} \{p_1 L_{\Pi_t(r)}\} \\ &\geqslant p_1 L_{\Pi_t(r)}. \end{aligned}$$

Hence, the minimum decrease in the objective function is

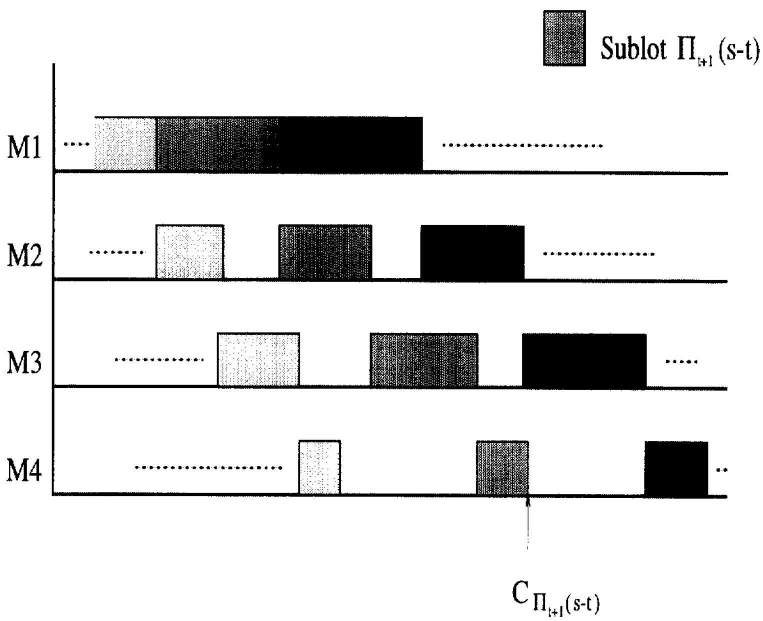

Fig. 7. Gantt chart of Π_{t-1}.

$$\Delta^- \geqslant p_1 L_{\Pi_t(r)} \sum_{l=r+1}^{s} L_{\Pi_t(l)}. \qquad \square$$

Result 6. *If the largest sublot among the first t sublots, $\Pi_t(r)$, is inserted $(s-t)$th place in the schedule (Fig. 7), then the maximum increase in the total flowtime Δ^+ is*

$$\Delta^+ \leqslant p_1 L_{\Pi_t(r)} \sum_{l=r+1}^{s} L_{\Pi_t(l)}.$$

Proof. Let C^+ be the increase in the completion time of removed sublot, when it is inserted in the $(s-t)$th place.

Observe that

$$\Delta^- = L_{\Pi_t(r)} C^+ + p_1 L_{\Pi_t(r)} \sum_{l=s-t+1}^{s} L_{\Pi_t(l)}.$$

C^+ can be written as

$$\begin{aligned} C^+ &\leqslant C_{i,\Pi_{t-1}(s-t)} - C_{i,\Pi_t(r)} \\ &\leqslant p_1 \sum_{l=1}^{s-t} L_{\Pi_t(l)} + L_{\Pi_t(r)} \sum_{v=2}^{m} p_v - \left(p_1 \sum_{l=1}^{r} L_{\Pi_t(l)} + L_{\Pi_t(r)} \sum_{v=2}^{m} p_v \right) \\ &\leqslant p_1 \sum_{l=r+1}^{s-t} L_{\Pi_t(l)}. \end{aligned}$$

Hence, the maximum increase is

$$\begin{aligned} \Delta^+ &\leqslant p_1 L_{\Pi_t(r)} \sum_{l=r+1}^{s-t} L_{\Pi_t(l)} + p_1 L_{\Pi_t(r)} \sum_{l=s-t+1}^{s} L_{\Pi_t(l)} \\ &\leqslant p_1 L_{\Pi_t(r)} \sum_{l=r+1}^{s} L_{\Pi_t(l)}. \qquad \square \end{aligned}$$

Therefore, in the tth step of proposed algorithm, the maximum overall increase in the mean flowtime value is

$$\Delta^+ - \Delta^- \leqslant 0.$$

Hence, the mean flowtime of the sublot schedule constructed by the algorithm is not worse. Thus, in the optimal solution, $L_k \leqslant L_{k+1}, \quad k = 1, \ldots, s$.

As shown in Fig. 7, the completion time of the sublot k on the last machine is,

$$C_{mk} = p_1 \sum_{l=1}^{k-1} L_l + L_k \sum_{i=1}^{m} p_i.$$

With this property the following concise formulation with a convex objective function and fewer constraints can be obtained;

$$\begin{aligned}
\min \quad & \sum_{k=1}^{s} L_k C_{mk} \\
\text{s.t.} \quad & C_{mk} - p_1 \sum_{l=1}^{k-1} L_l - L_k \sum_{i=1}^{m} p_i = 0, \quad k = 1,\ldots,s, \\
& \sum_{k=1}^{s} L_k = 1
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
\min \quad & \sum_{k=1}^{s} L_k \left(p_1 \sum_{l=1}^{k-1} L_l + L_k \sum_{i=1}^{m} p_i \right) \\
\text{s.t.} \quad & \sum_{k=1}^{s} L_k = 1.
\end{aligned}$$

Proof (*Result 4*). The Lagrangian function of the above problem is

$$\mathscr{L}(L_1,\ldots,L_s,\delta) = \sum_{k=1}^{s} L_k \left(p_1 \sum_{l=1}^{k-1} L_l + L_k \sum_{i=1}^{m} p_i \right) + \delta \left(\sum_{k=1}^{s} L_k - 1 \right),$$

then

$$\frac{\partial \mathscr{L}}{\partial L_r} = p_1 \sum_{l=1}^{s} L_l + 2L_r \sum_{i=1}^{m} p_i - p_1 L_r + \delta = 0$$

and

$$\frac{\partial \mathscr{L}}{\partial \delta} = \sum_{r=1}^{s} L_r - 1 = 0.$$

Since

$$\frac{\partial \mathscr{L}}{\partial L_r} - \frac{\partial \mathscr{L}}{\partial L_{r+1}} = 2(L_r - L_{r+1}) \sum_{i=1}^{m} p_i - p_1 L_r + p_1 L_{r+1} = 0$$

or

$$L_r \left(2 \sum_{i=1}^{m} p_i - p_1 \right) = L_{r+1} \left(2 \sum_{i=1}^{m} p_i - p_1 \right),$$

$$L_r = L_{r+1}.$$

But $\sum_{k=1}^{s} L_k = 1$ implies that $L_r = 1/s$ is the candidate optimal solution. However, to prove that it is the desired solution, we have to show that the objective function is convex. The Hessian matrix of the objective function is

$$\begin{bmatrix} a & b & b & b & b & \dots \\ b & a & b & b & b & \dots \\ b & b & a & b & b & \dots \\ b & b & b & a & b & \dots \\ b & b & b & b & a & \dots \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix},$$

where

$$a = 2\sum_{i=1}^{m} p_i \quad \text{and} \quad b = p_1.$$

The positive definiteness of the Hessian matrix will imply the convexity of the objective function. In order to prove that a matrix is positive definite, it is enough to show that the diagonal elements of the U matrix in LU decomposition of Hessian matrix (or, the pivot elements without row exchanges) are all positive.

Consider any matrix with above structure with $a > b \geqslant 0$. After the first Gaussian elimination step, we get the first pivot entry as $(a > 0)$, with updated matrix

$$\begin{bmatrix} a & b & b & b & b & \dots \\ 0 & \frac{a^2-b^2}{a} & \frac{a\cdot b-b^2}{a} & \frac{a\cdot b-b^2}{a} & \frac{a\cdot b-b^2}{a} & \dots \\ 0 & \frac{a\cdot b-b^2}{a} & \frac{a^2-b^2}{a} & \frac{a\cdot b-b^2}{a} & \frac{a\cdot b-b^2}{a} & \dots \\ 0 & \frac{a\cdot b-b^2}{a} & \frac{a\cdot b-b^2}{a} & \frac{a^2-b^2}{a} & \frac{a\cdot b-b^2}{a} & \dots \\ 0 & \frac{a\cdot b-b^2}{a} & \frac{a\cdot b-b^2}{a} & \frac{a\cdot b-b^2}{a} & \frac{a^2-b^2}{a} & \dots \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Since

$$\frac{a^2 - b^2}{a} > \frac{a \cdot b - b^2}{a} > 0$$

the sub-matrix starting from second row and second column has the same structure as the original one. Hence, its first pivot element will be positive and the resulting matrix will have the common structure. The proof follows inductively. □

A.2. Result 2

If $p_2/p_1 \equiv \pi > 1$, then optimal sublot sizes for the sublot completion model can be found by Algorithm 1.

Proof. Assume, without loss of generality, that $U = 1$ and the processing time of the job is 1 on the first machine and π on the second machine. Since $p_2 > p_1$, we have $\pi > 1$.

Result 7. *When $\pi > 1$, $\pi L_k \geqslant L_{k+1}$, $k = 1, \dots, s-1$, in an optimal schedule.*

Proof. Suppose the contrary, i.e., there exists an optimal solution $\bar{L} = (\bar{L}_1, \dots, \bar{L}_s)$ such that, at least for one k, $\pi\bar{L}_k < \bar{L}_{k+1}$. Let $v = \min_{1 \leqslant k \leqslant s-1}\{k \mid \pi\bar{L}_k < \bar{L}_{k+1}\}$. A new solution can be constructed for some $\epsilon > 0$, as

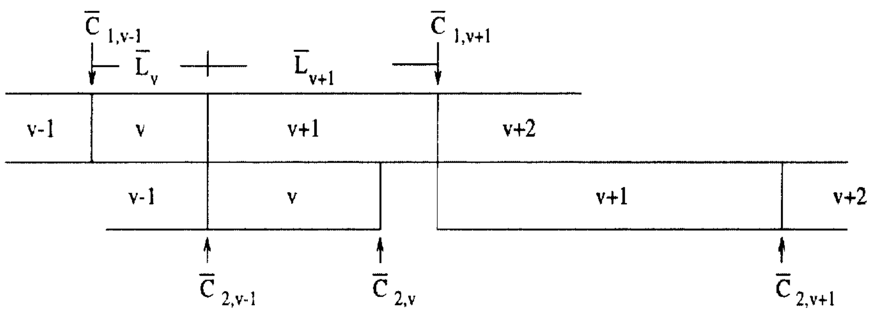

Fig. 8. Sublot completion, Case 1: $\overline{L} = (\overline{L}_1, \ldots, \overline{L}_v, \overline{L}_{v+1}, \ldots, \overline{L}_s)$.

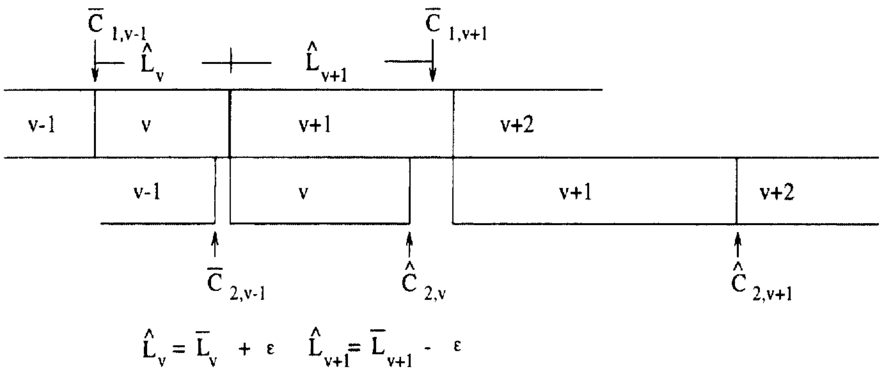

Fig. 9. Sublot completion, Case 1: $\hat{L} = (\overline{L}_1, \ldots, \overline{L}_v + \epsilon, \overline{L}_{v+1} - \epsilon, \ldots, \overline{L}_s)$.

$$\begin{aligned} \hat{L}_k &= \overline{L}_k, \qquad k = 1, \ldots, v-1, \\ \hat{L}_v &= \overline{L}_v + \epsilon, \\ \hat{L}_{v+1} &= \overline{L}_{v+1} - \epsilon, \\ \hat{L}_k &= \overline{L}_k, \qquad k = v+2, \ldots, s. \end{aligned}$$

It is sufficient to show that the new solution, $\hat{L}$ is feasible and $F(\hat{L}) < F(\overline{L})$.

Since $\sum_{k=1}^{v+1} \hat{L}_k = \sum_{k=1}^{v+1} \overline{L}_k$, and $\hat{C}_{1,v+1} = \overline{C}_{1,v+1} = \sum_{k=1}^{v+1} L_k$, for feasibility, it is enough to show

$$\hat{C}_{2,v+1} \leqslant \overline{C}_{2,v+1}. \tag{16}$$

We will now show that (16) holds and $F(\hat{L}) < F(\overline{L})$ for the following two possible cases.

Case 1: $\overline{C}_{1,v+1} > \overline{C}_{2,v}$ (see Figs. 8 and 9).

For small $\epsilon > 0$, we also have $\hat{C}_{1,v+1} > \hat{C}_{2,v}$.

$$\begin{aligned} \hat{C}_{2,v+1} &= \pi \hat{L}_{v+1} + \hat{C}_{1,v+1} \\ &= \pi(\overline{L}_{v+1} - \epsilon) + \overline{C}_{1,v+1} \\ &= \overline{C}_{2,v+1} - \pi\epsilon. \end{aligned}$$

Hence, Condition (16) is satisfied. To show $F(\hat{L}) < F(\overline{L})$, define $F_{v,v+1}(L)$ to be the contribution of sublots v and $v+1$ to the objective function and $U_{v,v+1} \equiv \overline{L}_v + \overline{L}_{v+1} = \hat{L}_v + \hat{L}_{v+1}$.

$$F_{v,v+1}(\bar{L}) = [\bar{C}_{1,v-1} + \bar{L}_v(1+\pi)]\bar{L}_v + [(\bar{C}_{1,v-1} + \bar{L}_v + \bar{L}_{v+1}) + \pi\bar{L}_{v+1}]\bar{L}_{v+1}$$
$$= (1+\pi)\bar{L}_v^2 + (U_{v,v+1} + \pi\bar{L}_{v+1})\bar{L}_{v+1} + \bar{C}_{1,v-1}U_{v,v+1}$$
$$= (1+\pi)\bar{L}_v^2 + U_{v,v+1}\bar{L}_{v+1} + \pi\bar{L}_{v+1}^2 + \bar{C}_{1,v-1}U_{v,v+1}.$$

Similarly,

$$F_{v,v+1}(\hat{L}) = (1+\pi)\hat{L}_v^2 + U_{v,v+1}\hat{L}_{v-1} + \pi\hat{L}_{v+1}^2 + \bar{C}_{1,v-1}U_{v,v+1}$$
$$= (1+\pi)(\bar{L}_v+\epsilon)^2 + U_{v,v+1}(\bar{L}_{v-1}-\epsilon) + \pi(\bar{L}_{v+1}-\epsilon)^2 + \bar{C}_{1,v-1}U_{v,v+1}$$

then,

$$F_{v,v+1}(\bar{L}) - F_{v,v-1}(\hat{L}) = -(2\pi+1)\epsilon^2 + (2\pi+1)(U_{v,v+1} - 2\bar{L}_v)\epsilon$$
$$= -(2\pi+1)\epsilon^2 + (2\pi+1)(\bar{L}_{v+1} - \bar{L}_v)\epsilon.$$

But, we know that $\pi\bar{L}_v < \bar{L}_{v+1}$, thus $\bar{L}_v < \bar{L}_{v+1}$, therefore it is clear that $F_{v,v+1}(\bar{L}) - F_{v,v+1}(\hat{L})$ is positive for some $\epsilon > 0$. Hence, $F(\bar{L}) > F(\hat{L})$, for some $\epsilon > 0$.

Case 2: $\bar{C}_{1,v-1} \leqslant \bar{C}_{2,v}$ (see Figs. 10 and 11).

For $\epsilon > 0$ we also have $\hat{C}_{1,v-1} \leqslant \hat{C}_{2,v}$,

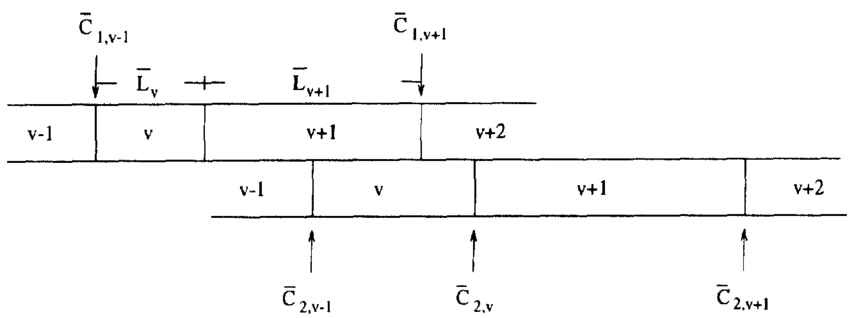

Fig. 10. Sublot completion, Case 2: $\bar{L} = (\bar{L}_1, \ldots, \bar{L}_v, \bar{L}_{v+1}, \ldots, \bar{L}_s)$.

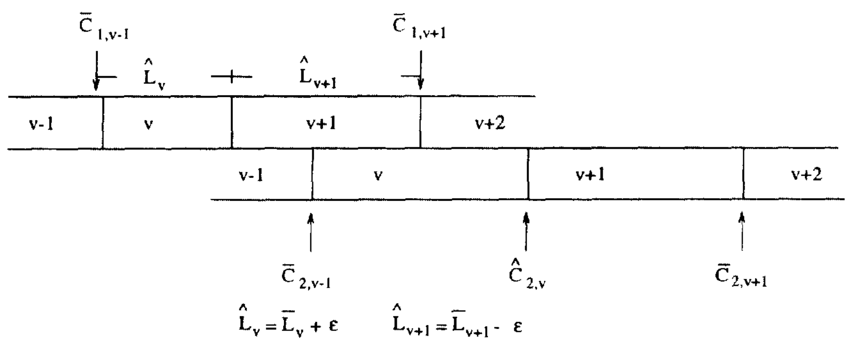

Fig. 11. Sublot completion, Case 2: $\hat{L} = (\bar{L}_1, \ldots, \bar{L}_v + \epsilon, \bar{L}_{v+1} - \epsilon, \ldots, \bar{L}_s)$.

$$\begin{aligned}\hat{C}_{2,v+1} &= \pi\hat{L}_{v+1} + \hat{C}_{2v} \\ &= \pi\hat{L}_{v+1} + \overline{C}_{2,v-1} + \pi\hat{L}_v \\ &= \pi\overline{L}_{v+1} + \overline{C}_{2,v-1} + \pi\overline{L}_v \\ &= \overline{C}_{2,v+1},\end{aligned}$$

since $\overline{L}_v + \overline{L}_{v+1} = \hat{L}_v + \hat{L}_{v+1}$. Hence, (16) is satisfied. To show $F(\hat{L}) < F(\overline{L})$

$$\begin{aligned}F_{v,v+1}(\overline{L}) &= (\overline{C}_{2,v-1} + \pi\overline{L}_v)\overline{L}_v + (\overline{C}_{2,v-1} + \pi U_{v,v+1})\overline{L}_{v+1} \\ &= \pi\overline{L}_v^2 + \pi U_{v,v+1}(U_{v,v+1} - \overline{L}_v) + U_{v,v+1}\overline{C}_{2,v-1}.\end{aligned}$$

Similarly

$$\begin{aligned}F_{v,v+1}(\hat{L}) &= \pi\hat{L}_v^2 + \pi U_{v,v+1}(U_{v,v+1} - \hat{L}_v) + U_{v,v+1}\overline{C}_{2,v-1} \\ &= \pi(\overline{L}_v + \epsilon)^2 + \pi U_{v,v+1}(U_{v,v+1} - \overline{L}_v - \epsilon) + U_{v,v+1}\overline{C}_{2,v-1}\end{aligned}$$

then

$$\begin{aligned}F_{v,v+1}(\overline{L}) - F_{v,v+1}(\hat{L}) &= -\pi\epsilon^2 + (U_{v,v+1} - 2\overline{L}_v)\pi\epsilon \\ &= -\pi\epsilon^2 + (\overline{L}_{v+1} - \overline{L}_v)\pi\epsilon.\end{aligned}$$

Since $(\overline{L}_{v+1} - \overline{L}_v)\pi$ is positive, $F_{v,v+1}(\overline{L}) - F_{v,v+1}(\hat{L})$ is positive for some $\epsilon > 0$. Thus, $F(\overline{L}) > F(\hat{L})$ for some $\epsilon > 0$. Thus in any optimal schedule, $\pi L_k \geqslant L_{k+1}$ $k = 1, \ldots, s-1$. □

Having observed that $\pi L_k \geqslant L_{k+1}$, $k = 1, \ldots, s-1$, for any optimal schedule, we can write the completion time of each sublot on the second machine as (Fig. 12)

$$C_{2k} = L_1 + \pi\sum_{\ell=1}^{k} L_\ell \quad k = 1, \ldots, s.$$

The mean flowtime is

$$\begin{aligned}F(L) &= \sum_{k=1}^{s} C_{2k}L_k \\ &= \sum_{k=1}^{s}(L_1 + \pi\sum_{\ell=1}^{k} L_\ell)L_k \\ &= L_1 + \pi\sum_{k=1}^{s}\sum_{\ell=1}^{k} L_\ell L_k.\end{aligned}$$

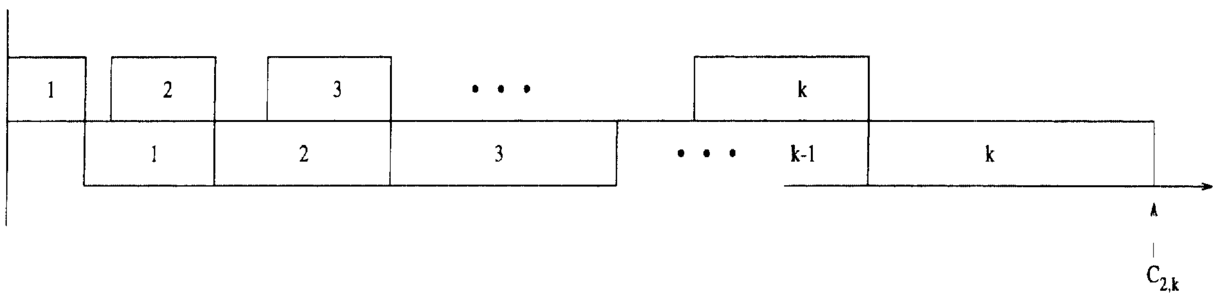

Fig. 12. Sublot completion, $\pi L_k \geqslant L_{k+1}$, $k = 1, \ldots, s$.

Then, an equivalent reformulation of the problem is

$$\min \quad F(L) = L_1 + \pi\sum\nolimits_{k=1}^{s}\sum\nolimits_{\ell=1}^{k} L_\ell L_k \tag{17}$$

$$\text{subject to} \quad \sum\nolimits_{k=1}^{s} L_k = 1, \tag{18}$$

$$L_{k+1} - \pi L_k \leqslant 0, \quad k = 1,\ldots,s-1, \tag{19}$$

$$L_k \geqslant 0, \quad k = 1,\ldots,s. \tag{20}$$

Result 8. *The following sublot sizes are optimal for (17)–(20),*

$$\overline{L}_1 = \frac{\frac{\pi^v-1}{\pi-1}\pi - (s-v)}{\frac{\pi^{2v}-1}{\pi^2-1}\pi(s-v) + \left(\frac{\pi^v-1}{\pi-1}\right)^2\pi}, \tag{21}$$

$$\overline{L}_k = \pi^{k-1}\overline{L}_1, \quad k = 1,\ldots,v, \tag{22}$$

$$\overline{L}_k = \frac{1 - \overline{L}_1\sum_{\ell=1}^{v}\pi^{\ell-1}}{(s-v)}, \quad k = v+1,\ldots,s \tag{23}$$

if $\pi\overline{L}_v \geqslant \overline{L}_{v+1} \geqslant \overline{L}_v$ *and* $v < s$.

Proof. The Gantt chart for an instance of the above sublot sizes will be as shown in Fig. 13. Since the objective function can be shown to be convex, similarly as in the previous case, it will be sufficient to show that the above solution is a Karush–Kuhn–Tucker point.

Assign, Lagrange multipliers δ for (18), and λ_k for (19). As seen in Fig. 13, only the first $v-1$ of the type (19) constraints are binding. So Karush–Kuhn–Tucker conditions for the solution are:
For L_1

$$1 + \pi + \pi L_1 + \delta - \pi\lambda_1 = 0. \tag{24}$$

For L_k, $k = 2,\ldots,v-1$

$$\pi + \pi L_k + \delta + \lambda_{k-1} - \pi\lambda_k = 0. \tag{25}$$

For L_v

$$\pi + \pi L_v + \delta + \lambda_{v-1} = 0. \tag{26}$$

For L_k, $k = v+1,\ldots,s$

$$\pi + \pi L_k + \delta = 0. \tag{27}$$

We have the following solution to system (24)–(27). Using the values $\overline{L} = (\overline{L}_1,\ldots,\overline{L}_s)$, and noting that $\overline{L}_k = \overline{L}_s$ $k = v+1,\ldots,s$, we get from (27),

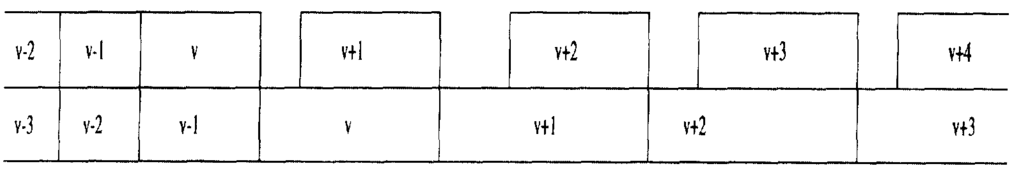

Fig. 13. Sublot completion, optimal sublots.

$$\delta = -\pi - \pi \overline{L}_s. \tag{28}$$

We also get from (26) and (28),

$$\lambda_{v-1} = \pi(\overline{L}_s - \overline{L}_v), \tag{29}$$

which is non-negative.

From (25) we obtain

$$\begin{aligned} \lambda_k &= \pi\lambda_{k+1} - \delta - \pi - \pi\overline{L}_{k+1}, \qquad k = 1, \ldots, v-2, \\ \lambda_k &= \pi\lambda_{k+1} + \pi(\overline{L}_s - \overline{L}_{k+1}), \quad k = 1, \ldots, v-2, \end{aligned} \tag{30}$$

which together with (29), proves the non-negativity of λ_k, $k = 1, \ldots, v-2$. We have from (29) and (30)

$$\lambda_1 = \left(\frac{\pi^v - \pi}{\pi - 1}\right)\overline{L}_s - \left(\frac{\pi^{2v} - \pi^2}{\pi^2 - 1}\right)\overline{L}_1. \tag{31}$$

On the other hand (24) and (28) give

$$\lambda_1 = \frac{1 + \pi\overline{L}_1 - \pi\overline{L}_s}{\pi}. \tag{32}$$

We also need to show that the sublot sizes result in a consistent solution of Lagrange multipliers

$$\lambda_1 = \frac{1 + \pi\overline{L}_1 - \pi\overline{L}_s}{\pi} = \left(\frac{\pi^v - \pi}{\pi - 1}\right)\overline{L}_s - \left(\frac{\pi^{2v} - \pi^2}{\pi^2 - 1}\right)\overline{L}_1,$$

$$\left(\frac{\pi^v - \pi}{\pi - 1}\right)\overline{L}_s - \left(\frac{\pi^{2v} - \pi^2}{\pi^2 - 1}\right)\overline{L}_1 - \overline{L}_1 + \overline{L}_s = \frac{1}{\pi},$$

$$\left(\frac{\pi^v - 1}{\pi - 1}\right)\overline{L}_s - \left(\frac{\pi^{2v} - 1}{\pi^2 - 1}\right)\overline{L}_1 = \frac{1}{\pi}.$$

Using (23)

$$\left(\frac{\pi^v - 1}{\pi - 1}\right)\frac{1 - \overline{L}_1(\frac{\pi^v - 1}{\pi - 1})}{(s - v)} - \left(\frac{\pi^{2v} - 1}{\pi^2 - 1}\right)\overline{L}_1 = \frac{1}{\pi}.$$

$$\left(\frac{\pi^v - 1}{\pi - 1}\right)\frac{1}{(s - v)} - \frac{1}{\pi} = \left(\frac{\pi^v - 1}{\pi - 1}\right)^2\frac{\overline{L}_1}{(s - v)} + \left(\frac{\pi^{2v} - 1}{\pi^2 - 1}\right)\overline{L}_1,$$

which results in

$$\overline{L}_1 = \frac{\frac{\pi^v - 1}{\pi - 1}\pi - (s - v)}{\frac{\pi^{2v} - 1}{\pi^2 - 1}\pi(s - v) + (\frac{\pi^v - 1}{\pi - 1})^2\pi}. \qquad \square$$

For $v = s$ (i.e. all the sublot sizes are geometric), we have the system of equations (24)–(26). The system has a consistent solution, hence it is enough only to show the non-negativity of the Lagrange multipliers, λ_k, $k = 1, \ldots, s-1$. We have

$$\lambda_{s-1} = \frac{-\pi^{2s} + 2\pi^{s+1} + 2\pi^s - 2\pi - 1}{(\pi^s - 1)(\pi + 1)\sum_{\ell=0}^{s-1}\pi^\ell}.$$

$$\lambda_k = \lambda_{s-1}\sum_{\ell=0}^{s-k-1}\pi^\ell + \sum_{\ell=k-1}^{s-1}(L_s - L_\ell)\pi^{\ell-k}.$$

Since $\lambda_k > \lambda_{s-1}$ $k = 1,\ldots,s-2$, it is sufficient to check the non-negativity of λ_{s-1}.

Hence, all the sublot sizes are geometric ($v = s$), only if the polynomial in the numerator of λ_{s-1} is positive for a given π, since the denominator is always positive.

Combining these results, the following algorithm, repeated for convenience, solves the problem:

Algorithm 1

$v \leftarrow 0$, optimal$\leftarrow$FALSE
If $f(\pi) = -\pi^{2s} + 2\pi^{s+1} + 2\pi^s - 2\pi - 1 > 0$
  optimal $\leftarrow$ TRUE, geometric sublots are optimal
While not optimal
  $v \leftarrow v + 1$
  $\overline{L}_1 \leftarrow [\frac{\pi^v - 1}{\pi - 1}\pi - (s - v)]/[\frac{\pi^{2v}-1}{\pi^2-1}\pi(s - v) + (\frac{\pi^v-1}{\pi-1})^2\pi]$
  $\overline{L}_k \leftarrow \pi^{k-1}\overline{L}_1,\ k = 2,\ldots,v$
  $\overline{L}_k \leftarrow [1 - \overline{L}_1\sum_{\ell=1}^{v}\pi^{\ell-1}]/[(s - v)],\ k = v + 1,\ldots,s$
  if $\pi\,\overline{L}_v \geqslant \overline{L}_{v+1} \geqslant \overline{L}_v$,
    optimal $\leftarrow$ TRUE, $\overline{L} = (\overline{L}_1,\ldots,\overline{L}_s)$ is optimal

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