The Inverted Pendulum

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Abstract

If an oscillating force is applied to an inverted pendulum it is possible to keep it completely vertical, that is $\pi/2$ degrees from the surface the pendulum lies on, with some stability depending on the frequency it oscillates at. This paper describes how and why this stability occurs for a particular inverted pendulum created by Mr. Jahan Sarashid.

1 Introduction

This paper is completely theocrical, except for a small calculation to test for approximate similarities with what has been observed. That is this paper does not try to summarize large data sets but tries to create an intuitively understandable model for the phenomenon at hand. Approaching this problem in a completely theoretical fashion is interesting and possibly rewarding in that the math used in explaining this phenomenon may show how this phenomenon relates to other systems that are completely different on the surface but are fundamentally similar.

A picture of the inverted pendulum under investigation is found in Figure 1. For this paper the simplification is made that the pendulum has a mass m at the end of a massless stick of length l. This simplification is acceptable for a rough approximation but may need to be refined for future calculations. The rod is attached to a vertically oscillating table; the position of which is given by $A\cos(\omega_s t)$, again a simplification is made that this source frequency,ω_s, is uniform. A mechanical description of the simplified pendulum is found in Figure 2 [1].

Thus the task at hand is to show that if ω_s is large enough, the pendulum will not fall over, and to find what big enough means. This paper is divided into 3 sections. The first section is this introduction. The second deals with the math model and the third tries to sum up the findings of the previous section and considers where thought should be put next to understand the phenomenon.

[1]This figure was inspired by the first page of [1]

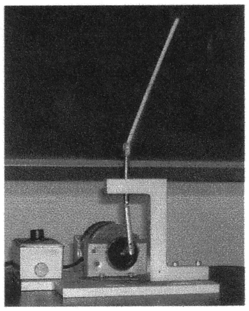

Figure 1: picture of Mr. Jahan Sarashid's inverted pendulum

2 Developing the Math Model

This section develops a simple but robust math model for modeling the behavior of the inverted pendulum.

2.1 The motion of the forcing agent

To understand the motion of the inverted pendulum it is quite appropriate to begin with getting a handle on the force that creates the oscillations in the first place. So using the Figure 2, geometry, and a little differentiation the motion of this body is gleaned in this section.[2]

[2]The remaining sections except for the Discussion are mainly your, Dr. Tahsiri's, work [1] put into my own words and equations. Some parts infact taken verbatim from your work. As such I hope I have correctly understood your instructions as I can claim almost no thought in this paper to be my own.

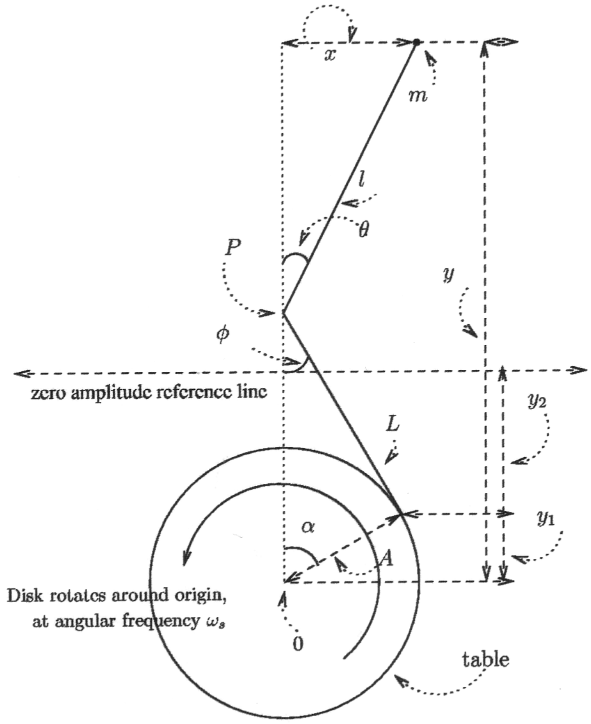

Figure 2: mechanical diagram of inverted pendulum.

$$y = y_1 + y_2$$
$$y_1 = A\cos\alpha$$
$$y_2 = L\cos\phi \Rightarrow$$

$$y = A\cos\alpha + L\cos\phi. \quad (1)$$

Also from Figure 2:

$$\sin\phi = \frac{A}{L}\sin\alpha \Rightarrow \cos^2\phi = 1 - \frac{A^2}{L^2}\sin^2\alpha$$

substituting the above into (1):

$$y = A\cos\alpha + L\sqrt{1 - \frac{A^2}{L^2}\sin^2\alpha}. \quad (2)$$

This fits into the intuitive definition of y as:

$$y(\alpha = 0) = L + A \text{ and,}$$

$$y(\alpha = \pi) = L - A$$

which is required observing Figure 2.

2.2 Finding the velocity and acceleration of point, P with respect to the center of the disk.

[3] If it is assumed that $\frac{A}{L} \ll 1$ then (2) can be expanded binomially and the 2nd term approximation is good:

$$y \approx A\cos\alpha + L(1 - \frac{1}{2}\frac{A^2}{L^2}\sin^2)$$
$$= A\cos\alpha + L - \frac{1}{2}\frac{A^2}{L}\sin^2\alpha$$

From the Figure:

$$\alpha = \omega_s t \Rightarrow$$

As it is assumed that ω_s is a constant this implies that $y = y(t)$.

$$y(t) \approx A\cos(\omega_s t) + L - \frac{1}{2}\frac{A^2}{L}\sin^2(\omega_s t) \quad (3)$$

Taking the derivative of $y(t)$ with respect to t produces:

$$v_y = \dot{y}(t) \approx -A\omega_s\sin(\omega_s t) - \frac{\omega_s A^2}{2L}\sin(2\omega_s t) \quad (4)$$

$$a_y = \ddot{y}(t) \approx -A{\omega_s}^2\cos(\omega_s t) - \frac{{\omega_s}^2 A^2}{L}\cos(2\omega_s t) \quad (5)$$

y is the vertical position of point P in relation to point 0 in Figure 2. The equations just solved are the necessary vertical relations between the position of the rotating body on the wheel that drives the pendulum and point P, the point at which the pendulum is attached. There is now enough work done to analyze the motion of the pendulum's rod.

2.3 Motion of the Inverted Pendulum

A method that works well for many systems, this one included, is using Lagrange's equations of motion. First the Lagrange,$\mathcal{L}$ is defined as the total kinetic energy of a system minus the total potential energy of the system:

$$\mathcal{L} = T - V \quad (6)$$

[3] You developed the pendulum's velocity for both 0 and 0′ however for the task at hand only the first is necessary.

and the equations of motion themselves are:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0. \tag{7}$$

where q_i represent indpendent generalized coordinates. For the purposes at hand it will be shown that $(q_1, q_2, \ldots)$ is simply (θ).

Now the total kinetic energy of the system is:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{Y}^2) \tag{8}$$

and the total potential energy of the sytsem, V is:

$$V = mgY \tag{9}$$

It is useful to get an intuitive feel for the movement of the pendulum. As such here is a simplified version of Figure 2.

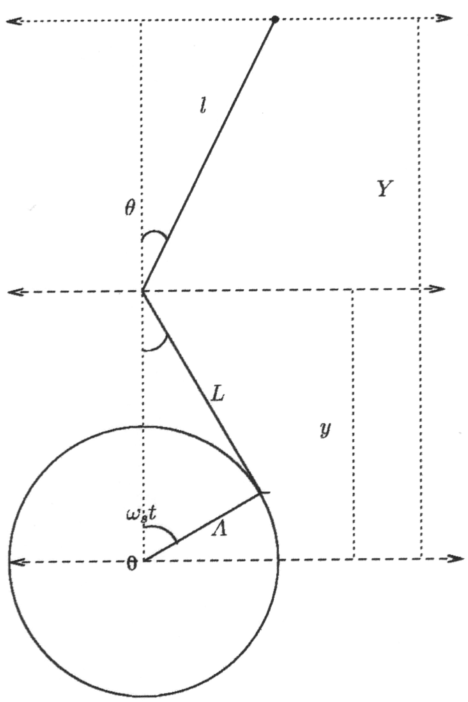

Figure 3: Simple diagram of inverted pendulum's motion.

From Figure 3 it is apparent that:

$$Y = y + l\cos\theta \tag{10}$$

using (4) this implies:

$$\dot{Y} \approx -A\omega_s \sin(\omega_s t) - \frac{\omega_s A^2}{2L}\sin(2\omega_s t) + (-l\sin\theta\dot{\theta}) \tag{11}$$

and the square is equal to:[4]

$$\begin{aligned}\dot{Y}^2 \approx{} & 2\,A\omega_s \sin(\omega_s t)\, l \sin(\theta(t))\,\dot{\theta}(t) \\ & + A^2{\omega_s}^2 (\sin(\omega_s t))^2 + l^2 (\sin(\theta(t)))^2 \left(\dot{\theta}(t)\right)^2 \\ & + l^2 (\sin(\theta(t)))^2 \left(\dot{\theta}(t)\right)^2 \\ & + \frac{2\,\omega_s A^2 \sin(\omega_s t)\cos(\omega_s t)\, l \sin(\theta(t))\,\dot{\theta}(t)}{L} \\ & + \frac{2\,A^3{\omega_s}^2 (\sin(\omega_s t))^2 \cos(\omega_s t)}{L} \\ & + \frac{{\omega_s}^2 A^4 (\sin(\omega_s t))^2 (\cos(\omega_s t))^2}{L^2}\end{aligned} \tag{12}$$

However as this paper is only interested in rough approximations and as $L \gg A$ the last three terms are very small (for small values of ω_s) compared to the other terms. Thus this paper restricts itself to small values of ω_s and the last three terms will be discarded. That is:

$$\begin{aligned}\dot{Y}^2 \approx{} & 2\,A\omega_s \sin(\omega_s t)\, l \sin(\theta(t))\,\dot{\theta}(t) \\ & + A^2{\omega_s}^2 (\sin(\omega_s t))^2 + l^2 (\sin(\theta(t)))^2 \left(\dot{\theta}(t)\right)^2 \\ & + l^2 (\sin(\theta(t)))^2 \left(\dot{\theta}(t)\right)^2\end{aligned} \tag{13}$$

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[4]I obtained the square using the following Maple code- although it would have been easy enough to do it by hand it saves some time and gives me a little experience:

```
>v[y]:=-A*omega[s]*sin(omega[s]*t)
-omega[s]*A^2*sin(2*omega[s]*t)/(2*L)
-l*sin(theta(t))*diff(theta(t),t);
```

$$-A\omega_s \sin(\omega_s t) - 1/2\,\frac{\omega_s A^2 \sin(2\,\omega_s t)}{L} - l \sin(\theta(t))\,\frac{d}{dt}\theta(t)$$

```
> collect(expand(v[y]^2),L);
```

$$\begin{aligned}& 2\,A\omega_s \sin(\omega_s t)\, l \sin(\theta(t))\,\frac{d}{dt}\theta(t) \\ & + A^2{\omega_s}^2 (\sin(\omega_s t))^2 + l^2 (\sin(\theta(t)))^2 \quad \frac{d}{dt}\theta(t) \quad ^2 \\ & + l^2 (\sin(\theta(t)))^2 \quad \frac{d}{dt}\theta(t) \quad ^2 \\ & + \frac{2\,\omega_s A^2 \sin(\omega_s t)\cos(\omega_s t)\, l \sin(\theta(t))\,\frac{d}{dt}\theta(t)\, 2 + A^3{\omega_s}^2 (\sin(\omega_s t))^2 \cos(\omega_s t)}{L} \\ & + \frac{{\omega_s}^2 A^4 (\sin(\omega_s t))^2 (\cos(\omega_s t))^2}{L^2}\end{aligned}$$

in the x direction:

$$x = l\sin\theta \Rightarrow \dot{x} = l\cos\theta\dot{\theta}. \tag{14}$$

So that:

$$\dot{x}^2 = l^2\cos^2\theta\dot{\theta}^2 \tag{15}$$

Note that the last term of $\dot{Y}^2$ combines with $\dot{x}^2$ So the kinetic energy is:

$$\begin{aligned} T \approx & \frac{1}{2}m(l^2\dot{\theta}^2 + 2\,A\omega_s \sin(\omega_s t)\, l\sin(\theta(t))\,\dot{\theta}(t) \\ & + A^2{\omega_s}^2(\sin(\omega_s t))^2 + l^2(\sin(\theta(t)))^2\left(\dot{\theta}(t)\right)^2) \end{aligned} \tag{16}$$

Using (3) and (10), and making the previous approximation that because $L \gg A$ the term with L in the denominator can be discarded, the potential energy for this system is:

$$V = mg(A\cos\omega_s t + L + l\cos\theta) \tag{17}$$

So

$$\begin{aligned} \mathcal{L} = & [\frac{1}{2}m(l^2\dot{\theta}^2 + 2\,A\omega_s \sin(\omega_s t)\, l\sin(\theta(t))\,\dot{\theta}(t) \\ & + A^2{\omega_s}^2(\sin(\omega_s t))^2 + l^2(\sin(\theta(t)))^2\left(\dot{\theta}(t)\right)^2] \\ & - mg(A\cos\omega_s t + l + l\cos\theta) \end{aligned} \tag{18}$$

As can be seen by this equation the only thing $\mathcal{L}$ is a function of is θ and t therefore the Lagrange equation is:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = 0. \tag{19}$$

Now,

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = & \\ & = ml\frac{d}{dt}(l\dot{\theta} + A\omega_s\sin(\theta)\sin(\omega_s t)) \\ & = ml(\ddot{\theta} + A\omega_s\cos(\theta)\dot{\theta}\sin(\omega_s t) \\ & + A\omega_s^2\cos(\omega_s t)\sin(\theta) \end{aligned} \tag{20}$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta} = mA\omega_s l\dot{\theta}\cos\theta\sin\omega t + mgl\sin\theta \tag{21}$$

So that the equation of motion is:

$$\begin{aligned} & ml(\ddot{\theta} + A\omega_s\cos(\theta)\dot{theta}\sin(\omega_s t) + \\ & A\omega_s^2\cos(\omega_s t)\sin(\theta) - mA\omega_s l\dot{\theta}\cos\theta\sin\omega t \\ & + mgl\sin\theta = 0 \end{aligned}$$

which is:

$$l^2\ddot{\theta} + A{\omega_s}^2\sin\theta\cos(\omega_s t) - g\sin\theta = 0 \tag{22}$$

Letting $\tau = \omega_s t$ and letting $\omega_0 = g/l$ the final ODE is obtained with respect to τ:

$$\ddot{\theta} + A/l{\omega_s}^2\sin\theta\cos\tau - (\frac{\omega_0}{\omega_s})^2\sin\theta = 0 \tag{23}$$

so the equation of motion for this problem is: Using the above equation of motion two plots of the angle θ of the rod in time have been plotted in Figure 4 and 5. Figure 4 represents a stable- i.e. ω_s is large enough- situation, and Figure 5 depicts the nature of the pendulum's movement if ω_s is too small too remain constantly vertical.

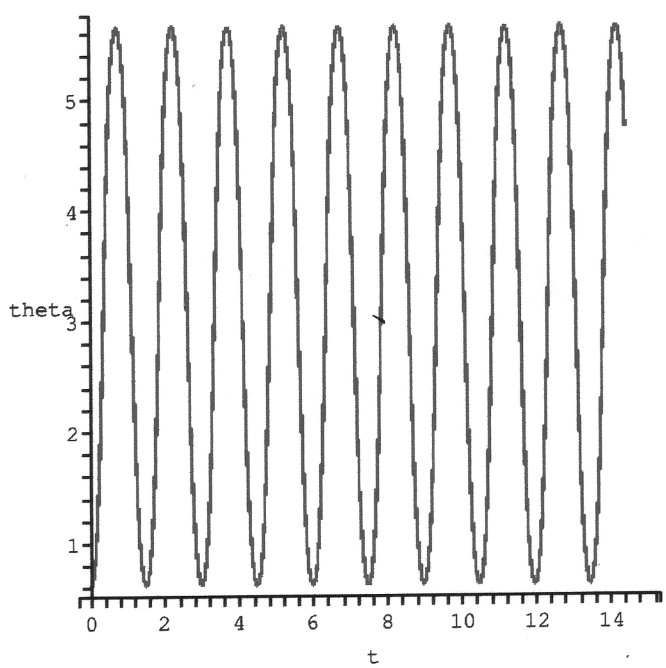

Figure 4: Stable Theoretical Inverted Pendulum with $l = .20m, g = 9.8\frac{m}{s^2}, A = 0.0013m, \omega = 200Hz * rad$

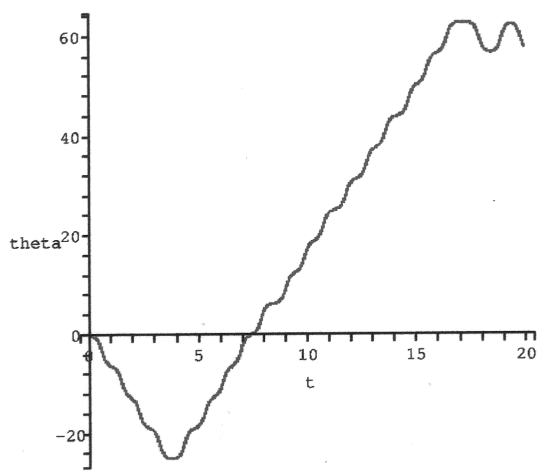

Figure 5: Unstable Theoretical Inverted Pendulum with $l = .20m, g = 9.8\frac{m}{s^2}, A = 0.0013m, \omega = 20 * Hzrad$

3 Discussion

With a little work it has been shown that an inverted pendulum can be modeled reasonbly well with not too much difficulty. It is of interest to note the similarity between the simplified inverted pendulum's ODE and that of the damped pendulum:

$$ml\ddot{\theta} + \gamma\dot{\theta} + mg\sin\theta = 0. \tag{24}$$

It is understandable as one can think of Earth's gravitational field operating in just the opposite way for both pendulums but with the same strength.

The approximations and simplifications that were made to create a simple enough model to work with is not totally acceptable for at large values of ω_s the terms neglected will have an impact. Also while $L > A$,L is not of the magnitude larger that a two term binomial approximation for y_2 is totally acceptable and a better approximation should possibly be taken (depending on the needs of the experimentor). With the use of a computer and a symbolic calculator (such as Mathematica or Maple) it seems that a more precise estimate of an inverted pendulum's behavior is relatively straight forward and possible with a little thought and work. This will be left to another paper and most probably another person altogether however.

"Thank you"s and Acknowledgments

Thank you Dr. Jorgensen for making what is imperfect more perfect, i.e. reading and critiquing this paper. Thank you Dr. Tahsiri for this chance to consider what happens when you flip the world - or at least part of it- upside down.

References

[1] H. Tahsiri: *Final Exam Project: Inverted Pendulum,* Physics 560A CSULB handout Fall 2004.

[2] Jorgensen, Banks, Hyatt, and Shleyzer; *Teaching the Nonlinear Pendulum,* The Physics Teacher 32, 4 (1994).

Physics 560A

Instructor: H. Tahsiri

The goals of this course are for you to become more adept at applying mathematical tools to physics problems and to be better prepared for other upper or graduate division courses and the real world. One of the tools we will spend quite *a* bit of time on is *Mathematica* powerful software that can solve practically any math problem provided you can express it correctly (it is quite demanding) and interpret the results (they are not always obvious). To make this easier we will meet in *a* computer lab, We will meet every Tues and Thurs during lecture.

Textbook : There is no required text for this course. Any good advanced book on mathematical physics will be adequate. Appropriate texts are by Mathew and Walker and Arfken.

Required Text : *Mathematica*, Schaum ' *s* Outlines, $16 .95

Grading : Your grade will be based on the total number of points obtained from homework, take home research project midterm exam and the take home research project final exam. I'll assist you with your projects.

Material covered :The exact material covered usually depends upon each class's interest and abilities; however, the basics will include: Analytical and numerical solution to linear and nonlinear differential equations, Integrals, Complex variables, Fourier and Laplace Transform. Analytical / numerical solutions to partial differential equations of types "Parabolic", "Hyperbolic" and "Elliptic".

$$L = \frac{1}{2} m[l^2\dot{\theta}^2 + A^2\omega_s^2 \sin^2(\omega_s t) + 2A\omega_s l\dot{\theta}\sin(\theta)\sin(\omega_s t)] - mg[A\cos(\omega_s t) + l\cos(\theta)] \quad (16)$$

4. Equation of Motion

We are now ready to obtain the equation of motion for the inverted pendulum system. The Lagrange's equation[3] is written as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad (17)$$

We evaluate **Eq.17** term by term. The first term is given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{1}{2} m[2l^2\ddot{\theta} + 2A\omega_s l\dot{\theta}\cos(\theta)\sin(\omega_s t) + 2A\omega_s^2 l\sin(\theta)\cos(\omega_s t) \quad (18)$$

The second term is given by,

$$\frac{\partial L}{\partial \theta} = mA\omega_s l\dot{\theta}\cos(\theta)\sin(\omega_s t) + mgl\sin(\theta) \quad (19)$$

Putting **Eq.18** and **Eq.19** together, the Lagrange's equation becomes,

$$l\frac{d^2\theta}{dt^2} + [A\omega_s^2\cos(\omega_s t) - g]\sin(\theta) = 0 \quad (20)$$

This is the equation of motion for our 2-dimensional inverted pendulum. This equation has no analytical result. However, it can be solved numerically using Mathematica. If we specify some values to A, l, g, and ω_s such that the motion is stable (stability analysis is in the next session), we are able to obtain a numerical solution to **Eq.20**. **Figure 3** shows a plot to the solution.

Since Eq.20 gives no analytical result, we will perform a series of substitution and approximation to reduce **Eq.20** to an analytically solvable equation.

If we define

$$\tau = \omega_s t \quad (21)$$

and

$$\omega_0 = \sqrt{\frac{g}{l}} \quad (22)$$

Absorbing **Eq.21** and **Eq.22**, and after a little arrangement, **Eq.20** becomes,

$$\frac{d^2\theta}{d\tau^2} + [\frac{A}{l}\cos(\tau) - \frac{\omega_0^2}{\omega_s^2}]\sin(\theta) = 0 \quad (23)$$

If we restrict θ to be small, we can make a small angle approximation $sin(\theta) = \theta$, then **Eq.23** becomes,

$$\frac{d^2\theta}{d\tau^2} + [\frac{A}{l}\cos(\tau) - \frac{\omega_0^2}{\omega_s^2}]\theta = 0 \quad (24)$$

It can be further simplified by replacing $cos(\tau)$ with its time average. Since the time average of $cos^2(\tau)$ is ½, then,

$$[\cos(\tau)]_{avg} = \frac{1}{\sqrt{2}} \quad (25)$$

Thus, **Eq.24** becomes,

$$\frac{d^2\theta}{d\tau^2} + [\frac{A}{\sqrt{2}l} - \frac{\omega_0^2}{\omega_s^2}]\theta = 0 \quad (26)$$

It is a second order linear differential equation with constant coefficients, which has the same form as **Eq.1**.

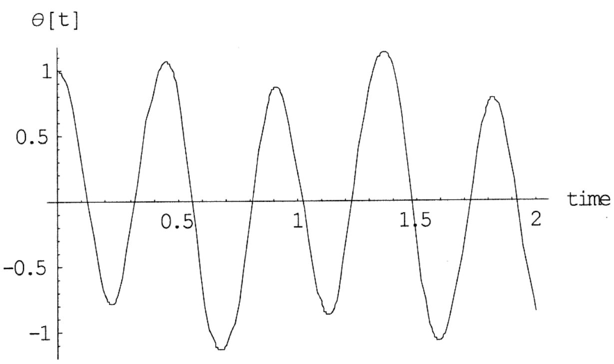

Figure 3 Plot of a stable solution to Eq.20 by Mathematica

3. The formulation of the Lagrange's equation can be found in Goldstein

5. Stability Analysis

In this session, we want to examine the stability of the inverted pendulum by studying the driving frequency, ω_s. The stability of the solution to **Eq.26** is summarized below:

(i) Stable Simple Harmonic Motion

$$\frac{A}{\sqrt{2}l} - \frac{\omega_0^2}{\omega_s^2} > 0 \qquad (27)$$

or

$$\omega_s^2 > \frac{\sqrt{2}l}{A}\omega_0^2 \qquad (28)$$

(ii) Unstable (falling)

$$\frac{A}{\sqrt{2}l} - \frac{\omega_0^2}{\omega_s^2} < 0 \qquad (29)$$

or

$$\omega_s^2 < \frac{\sqrt{2}l}{A}\omega_0^2 \qquad (30)$$

So with **Eq.22**, the critical driven frequency can now be expressed as,

$$\omega_c^2 = \frac{\sqrt{2}l}{A}\omega_0^2 = \frac{\sqrt{2}g}{A} \qquad (31)$$

One of the parameters that the critical driving frequency depends on is the radius of the rotating wheel, A. As shown in the **Eq.31**, ω_s^2 and A are inversely proportional. For instance, a larger disk, thus a larger A, requires a lower rotating frequency to keep the inverted pendulum stable. However, a larger A requires a longer connecting rod (larger L), since we assumed that $L >> A$ when we made the claim that the motion point **P** is a simple harmonic motion, as in **Eq.7**.

The other parameter is the acceleration due to gravity, g. ω_s^2 and g have a proportional relation. With, say, a higher g, a higher driving frequency is required to keep the motion stable, or in other words, to keep the mass from falling off the top.

Acknowledgment is made to Dr. Tahsiri the advisor to this paper. Acknowledgment is also made to Mr. Jahan Sarashid who made the inverted pendulum device (Figure 1). Special thank to Dr. George who spend his time to review this paper.

Two-Dimensional Inverted Pendulum

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1. Introduction

A regular ideal string-mass pendulum is well studied. Ignoring the force due to air friction, the only force acting on the string-mass system is the gravitational force on the mass. Newton's equation then reads,

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0 \qquad (1)$$

where θ is the angle between the string and the vertical, g is the acceleration due to gravitation, and l is the length of the string.

The small angle approximation solution, with $\sin\theta \cong \theta$, to this equation is,

$$\theta(t)=\theta_m\cos(\omega_0 t) \qquad (2)$$

with

$$\omega_0=\sqrt{\frac{g}{l}}$$

The idea of Inverted Pendulum was invented by Hooshang Tahsiri at UC Irvine and the mechanical device was made by Jahan Sarashid, as shown in **Figure 1**. In the following discussion, we will investigate the theoretical foundations underlying the motion of an inverted pendulum. We will restrict ourselves in the 2-dimensional case.

2. Inverted Pendulum – The Basic Idea

There are two forces acting on the rod-mass system of an inverted pendulum (assume the mass of the rod is negligible). **(1)** The gravitational force acting on the mass; **(2)** A vertical driving force acting on the lower end of the rod, point **P** in **Figure 2**. The gravitational force is simply a downward constant force. We will not discuss it here.

For the vertical driving force, we have no idea on the magnitude of the force. However, we can trace out the motion of the lower end of the rod (point **P**). **P** is connected to a fixed point **Q** on the rim of the rotating wheel with radius A by a rigid massless rod. The wheel is rotated at a constant angular frequency, ω_s, by an electrical motor. Since the motion of **P** is restricted in the vertical dimension, as we will see in the next section, it is *approximately* a simple harmonic motion, given by,

$$y_p(t)\cong A\cos\alpha \equiv A\cos(\omega_s t) \qquad (3)$$

Our aim is to find an analytical expression for the driving angular frequency, ω_s, that can keep the system stable, or not falling.

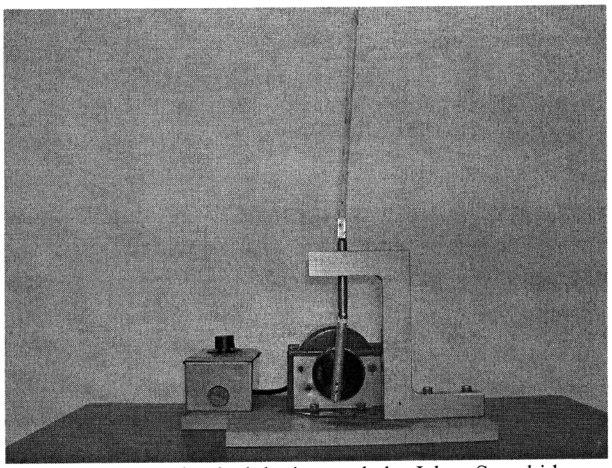

Figure 1 The mechanical device made by Jahan Sarashid

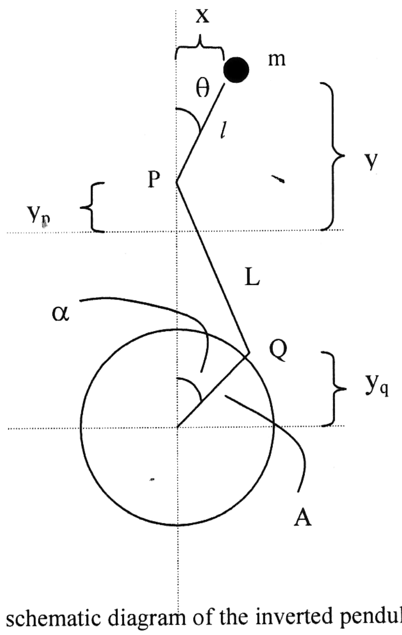

Figure 2 A schematic diagram of the inverted pendulum

3. Energy (Lagrangian) Analysis

Since finding an expression for the vertical force is not easy, instead of using Newton's equation, which emphasizes on the net force acting on the system, we will obtain the equation of motion by writing the Lagrangian of the system, which describes the system from the energy stand point of view. The Lagrangian is given by[1],

$$L = T - V \tag{4}$$

where T is the total kinetic energy and V is the total potential energy in our system.
Since V and T depend on the position of the mass (recall that the rod is massless) and its first derivative, respectively, we need to find expressions for them before we can write T and V and thus the Lagrangian.

(1) Vertical Position
The vertical position is given by,

$$y(t) = y_p(t) + l\cos\theta \tag{5}$$

where $y_p(t)$, the position of point **P**, is given by **Eq.3**.

Let us now verify **Eq.3**. **P** is connected, by a rigid rod, to a fixed point **Q** on the rim of the wheel rotating at a constant angular frequency, ω_s. So we know that the motion of **P** is restricted by the motion of **Q**. It has been studied that for a point moving on a circle with constant velocity, the Cartesian components (in our case, the vertical component) of its position is undergoing simple harmonic motion, with an angular frequency equals that of the circular motion of the point[2]. So, with $y_q = 0$ at the center of the wheel,

$$y_q(t) = A\cos\alpha \equiv A\cos(\omega_s t) \tag{6}$$

If we keep the connecting rod to be vertical all the time and let point **P** to move in a circle, the motion of **P** would be exactly the same as the motion of **Q**.

1. Discussion on the Lagrangian can be found in any undergraduate Classical Mechanics textbook. For instance, Goldstein.
2. Any introductory college Physics textbook. For instance, Young.

However, since the motion of **P** is restricted to be pure vertical, $y_p \neq y_q$. Nonetheless, y_q is a good approximation to y_p if L >> A. Then,

$$y_p(t) = y_q(t) = A\cos(\omega_s t) \tag{7}$$

And the vertical position of the mass is then given by,

$$y(t) = A\cos(\omega_s t) + l\cos\theta \tag{8}$$

Consequently, its first derivative is given by,

$$\frac{dy}{dt} = -(\omega_s A\sin(\omega_s t) + l\sin\theta\frac{d\theta}{dt}) \tag{9}$$

(2) Horizontal Position
The horizontal position is given by,

$$x(t) = l\sin\theta \tag{10}$$

Thus, its first derivative is,

$$\frac{dx}{dt} = l\cos\theta\frac{d\theta}{dt} \tag{11}$$

(3) Kinetic Energy
We are now ready to write the Lagrangian of the system. The total kinetic energy is given by,

$$T = \frac{1}{2}m((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2) \tag{12}$$

Substitute **Eq.9** and **Eq.11**, it becomes,

$$T = \frac{1}{2}m[l^2\dot{\theta}^2 + A^2\omega_s^2\sin^2(\omega_s t) + 2A\omega_s l\dot{\theta}\sin(\theta)\sin(\omega_s t)] \tag{13}$$

(4) Potential Energy
The potential energy is given by,

$$V = mgy(t) \tag{14}$$

Substitute **Eq.8**, it becomes,

$$V = mg[A\cos(\omega_s t) + l\cos(\theta)] \tag{15}$$

Putting Eq.13 and Eq.15 together, we obtain the Lagrangian,