Math 119B Spring 2007 Newberger

Worksheet: Separable Differential Equations

Use the power rule: $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$ for $n \neq -1$.

1. Find the general solutions for the following separable differential equations.

a. $y^2 \dfrac{dy}{dx} = 6x - x^3$

- Separate the variables.

$$y^2 dy = (6x - x^3)dx$$

- Integrate.

$$\int y^2 dy = \int (6x - x^3)dx \qquad \text{integrate using the power rule}$$

$$\frac{1}{3}y^3 + C_1 = 6(\frac{1}{2}x^2) - \frac{1}{4}x^4 + C_2 \qquad \text{subtract } C_1 \text{ from both sides}$$

$$\frac{1}{3}y^3 = 3x^2 - \frac{1}{4}x^4 + C_2 - C_1$$

Let $C = C_2 - C_1$. Then the equation becomes

$$\frac{1}{3}y^3 = 3x^2 - \frac{1}{4}x^4 + C.$$

- If you are asked to find a particular solution, the next step would be to use the additional information to solve for C. Since we are finding the general solution in this problem, our answer will include the variable C.
- Solve for y.

$$\frac{1}{3}y * 3 = 3x^2 - \frac{1}{4}x^4 + C \qquad \text{multiply both sides of the equation by 3}$$

$$3(\frac{1}{3}y * 3) = 3(3x^2 - \frac{1}{4}x^4 + C) \qquad \text{simplify}$$

$$y * 3 = 9x^2 - \frac{3}{4}x^4 + 3C \qquad \text{take the cube root of both sides}$$

$$y = (9x^2 - \frac{3}{4}x^4 + 3C)^{1/3}$$

So the general solution to this differential equation is

$$y = (9x^2 - \frac{3}{4}x^4 + 3C)^{1/3}$$

where C is any real number.

b. $\dfrac{dy}{dx} = 5x^{2/3}y^4$

- Separate the variables.

$$y^{-4}dy = 5x^{2/3}dx$$

- Integrate.

$$\int y^{-4}dy = \int 5x^{2/3}dx \qquad \text{integrate using the power rule}$$
$$-\frac{1}{3}y^{-3} + C_1 = 5(\frac{3}{5}x^{5/3}) + C_2 \qquad \text{subtract } C_1 \text{ from both sides}$$
$$-\frac{1}{3}y^{-3} = 3x^{5/3} + C_2 - C_1$$

Let $C = C_2 - C_1$. Then the equation becomes

$$-\frac{1}{3}y^{-3} = 3x^{5/3} + C.$$

- If you are asked to find a particular solution, the next step would be to use the additional information to solve for C. Since we are finding the general solution in this problem, our answer will include the variable C.
- Solve for y.

$$-\frac{1}{3}y^{-3} = 3x^{5/3} + C \qquad \text{multiply both sides of the equation by -3}$$
$$-3(-\frac{1}{3}y^{-3}) = -3(3x^{5/3} + C) \qquad \text{simplify}$$
$$y^{-3} = -9x^{5/3} - 3C \qquad \text{raise both sides to -1/3}$$
$$(y^{-3})^{-1/3} = (-9x^{5/3} - 3C)^{-1/3} \qquad \text{use the law of exponents } (y^m)^n = y^{mn}$$
$$y = (-9x^{5/3} - 3C)^{-1/3}$$

So the general solution to this differential equation is

$$y = (-9x^{5/3} - 3C)^{-1/3}$$

where C is any real number.

c. $\dfrac{dy}{dx} = \sqrt{x}e^{4y}$

Use the formula $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$

- Separate the variables.

$$\frac{1}{e^{4y}}dy = \sqrt{x}dx$$

We want to use the formulas $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$ and $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$, so we need to change the square root signs into fractional exponents, and use the laws of exponents to move the e^{4x} into the numerator.

$$e^{-4y}dy = x^{1/2}dx$$

- Integrate.

$$\int e^{-4y}dy = \int x^{1/2}dx$$
$$\frac{1}{-4}e^{-4y} + C_1 = \frac{2}{3}x^{3/2} + C_2 \qquad \text{subtract } C_1 \text{ from both sides}$$
$$\frac{1}{-4}e^{-4y} = \frac{2}{3}x^{3/2} + C_2 - C_1$$

Let $C = C_2 - C_1$. Then the equation becomes

$$\frac{1}{-4}e^{-4y} = \frac{2}{3}x^{3/2} + C$$

- If you are asked to find a particular solution, the next step would be to use the additional information to solve for C. Since we are finding the general solution in this problem, our answer will include the variable C.
- Solve for y.

$$\frac{1}{-4}e^{-4y} = \frac{2}{3}x^{3/2} + C \qquad \text{multiply both sides of the equation by -4}$$
$$-4(\frac{1}{-4}e^{-4y}) = -4(\frac{2}{3}x^{3/2} + C) \qquad \text{simplify}$$
$$e^{-4y} = \frac{-8}{3}x^{3/2} - 4C \qquad \text{take the natural log (ln) of both sides}$$
$$-4y = \ln(\frac{-8}{3}x^{3/2} - 4C) \qquad \text{divide both sides of the equation by -4}$$
$$y = \frac{\ln(\frac{-8}{3}x^{3/2} - 4C)}{-4}$$

So the general solution to this differential equation is

$$y = \frac{\ln(\frac{-8}{3}x^{3/2} - 4C)}{-4}$$

where C is any real number.

d. $\dfrac{dy}{dx} = e^{x+y}$

- Separate the variables.
  First, we will use the law of exponents that says $a^{m+n} = a^m a^n$.

$$\frac{dy}{dx} = e^x e^y$$

Now separate the variables.

$$\frac{dy}{e^y} = e^x dx$$

We want to use the formula $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$, so we use the laws of exponents to move the e^y into the numerator.

$$e^{-y}dy = e^x dx$$

- Integrate.

$$\int e^{-y} dy = \int e^x dx$$

$$\frac{1}{-1} e^{-y} + C_1 = e^x + C_2 \qquad \text{subtract } C_1 \text{ from both sides}$$

$$-e^{-y} = e^x + C_2 - C_1$$

Let $C = C_2 - C_1$. Then the equation becomes

$$-e^{-y} = e^x + C$$

- If you are asked to find a particular solution, the next step would be to use the additional information to solve for C. Since we are finding the general solution in this problem, our answer will include the variable C.
- Solve for y.

$$-e^{-y} = e^x + C \qquad \text{multiply both sides of the equation by -1}$$

$$e^{-y}) = -e^x - C \qquad \text{take the natural log (ln) of both sides}$$

$$-y = \ln(-e^x - C) \qquad \text{multiply both sides of the equation by -1}$$

$$y = -\ln(-e^x - C)$$

So the general solution to this differential equation is

$$y = -\ln(-e^x - C)$$

where C is any real number.

2. Find the particular solutions for the following separable differential equations. (These are the equations from (1a) and (1b) above.)

Use the given values of x and y to solve for the constant.

a. $y^2 \dfrac{dy}{dx} = 6x - x^3$ and $y(0) = 3$.

In part (1a) above, we found

$$\frac{1}{3} y^3 = 3x^2 - \frac{1}{4} x^4 + C.$$

Substitute $x = 0$ and $y = 4$, and solve for C.

$$\frac{1}{3} 4^3 = 3(0)^2 - \frac{1}{4}(0)^4 + C$$

$$\frac{1}{3} 4^3 = C$$

$$C = 256/3$$

In part (1a) above, we found

$$y = (9x^2 - \frac{3}{4} x^4 + 3C)^{1/3},$$

so the particular solution to this differential equation with $y(0) = 4$ is

$$y = (9x^2 - \frac{3}{4} x^4 + 256)^{1/3}$$

b. $\dfrac{dy}{dx} = 5x^{2/3}y^4$ and $y(1) = 8$.

In part (1b) above, we found

$$-\frac{1}{3}y^{-3} = 3x^{5/3} + C.$$

Substitute $x = 1$ and $y = 8$, and solve for C.

$$\begin{aligned} -\frac{1}{3}8^{-3} &= 3(1)^{5/3} + C \\ -1/1536 &= 3 + C \\ C &= -3 - 1/1536 = -3.00065 \end{aligned}$$

In part (1b) above, we found

$$y = (-9x^{5/3} - 3C)^{-1/3}$$

so the particular solution to this differential equation with $y(1) = 8$ is

$$y = (-9x^{5/3} + 9.0020)^{-1/3}$$

3. Find the following antiderivatives.

a. $\displaystyle\int \frac{dy}{4y} = \frac{1}{4}\int y^{-1}dy = \frac{1}{4}\ln|y| + C$

Use the formula $\int x^{-1}dx = \ln|x| + C$.

b. $\displaystyle\int \frac{dy}{1-10y}$

Let $u = 1 - 10y$.
Then $du = -dy$, so $dy = -du$.
Substitute these into the integral.

$$\begin{aligned} \int \frac{dy}{1-10y} &= \int \frac{-du}{u} \\ &= -\int \frac{du}{u} \\ &= -\ln|u| + C \qquad \text{put the } x\text{'s back in} \\ &= -\ln|1-10x| + C \end{aligned}$$

4. Find the general solutions for the following separable differential equations.

a. $\dfrac{dy}{dx} = xy$

- Separate the variables.

$$\frac{dy}{y} = x dx$$

- Integrate.

$$\begin{aligned}\int \frac{dy}{y} &= \int x dx \\ \ln|y| + C_1 &= \frac{1}{2}x^2 + C_2 && \text{subtract } C_1 \text{ from both sides} \\ \ln|y| &= \frac{1}{2}x^2 + C_2 - C_1\end{aligned}$$

  Let $C = C_2 - C_1$. Then the equation becomes

$$\ln|y| = \frac{1}{2}x^2 + C.$$

- If you are asked to find a particular solution, the next step would be to use the additional information to solve for C. Since we are finding the general solution in this problem, our answer will include the variable C.
- Solve for y.

$$\begin{aligned}\ln|y| &= \frac{1}{2}x^2 + C && \text{apply the exponential function to both sides} \\ e^{\ln|y|} &= e^{\frac{1}{2}x^2+C} && e^x \text{ and } \ln(x) \text{ are inverses of each other} \\ |y| &= e^{\frac{1}{2}x^2+C}.\end{aligned}$$

  This means that either $y = e^{\frac{1}{2}x^2+C}$ or $y = -e^{\frac{1}{2}x^2+C}$. (Right? If you knew that $|y| = 2$, then you'd know that y is either 2 or -2.)

When you write the general solution for this, give both solutions. Say, "The general solution is $y = e^{\frac{1}{2}x^2+C}$ or $y = -e^{\frac{1}{2}x^2+C}$.

b. $\dfrac{dy}{dx} = 3x - 5xy$

- Separate the variables.
  In order to get all the x's to one side, factor out an x from the right hand side.

$$\frac{dy}{dx} = x(3 - 5y)$$

  Now we can separate the variables

$$\frac{dy}{3 - 5y} = x dx$$

- Integrate.

$$\int \frac{dy}{3 - 5y} = \int x dx$$

To calculate the integral on the left, substitute $u = 3 - 5y$. Then $du = -5dy$ and we have $dy = \frac{1}{-5}du$.

$$\begin{aligned}
\int \frac{dy}{3-5y} &= \int x dx \\
\int \frac{\frac{1}{-5}du}{u} &= \frac{1}{2}x^2 + C_2 \\
\frac{1}{-5}\int \frac{du}{u} &= \frac{1}{2}x^2 + C_2 \\
\ln|u| + C_1 &= \frac{1}{2}x^2 + C_2 && \text{put the } y\text{'s back} \\
\ln|3-5y| + C_1 &= \frac{1}{2}x^2 + C_2 && \text{subtract } C_1 \text{ from both sides} \\
\ln|3-5y| &= \frac{1}{2}x^2 + C_2 - C_1
\end{aligned}$$

Let $C = C_2 - C_1$. Then the equation becomes

$$\ln|3-5y| = \frac{1}{2}x^2 + C.$$

- If you are asked to find a particular solution, the next step would be to use the additional information to solve for C. Since we are finding the general solution in this problem, our answer will include the variable C.
- Solve for y.

$$\begin{aligned}
\ln|3-5y| &= \frac{1}{2}x^2 + C && \text{apply the exponential function to both sides} \\
e^{\ln|3-5y|} &= e^{\frac{1}{2}x^2+C} && e^x \text{ and } \ln(x) \text{ are inverses of each other} \\
|3-5y| &= e^{\frac{1}{2}x^2+C}.
\end{aligned}$$

This means that either $3-5y = e^{\frac{1}{2}x^2+C}$ or $3-5y = -e^{\frac{1}{2}x^2+C}$. So we get either

$$y = \frac{e^{\frac{1}{2}x^2+C} - 3}{-5}$$

or

$$y = \frac{-e^{\frac{1}{2}x^2+C} - 3}{-5} = \frac{e^{\frac{1}{2}x^2+C} + 3}{5}.$$

The general solution is either either

$$y = \frac{e^{\frac{1}{2}x^2+C} - 3}{-5}$$

or

$$y = \frac{e^{\frac{1}{2}x^2+C} + 3}{5}.$$

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5. Find the particular solution to the equation
$$\frac{dy}{dx} = \frac{y-3}{2x+1},$$
if $y = 4$ when $x = 0$.

- Separate the variables.
$$\frac{dy}{y-3} = \frac{dx}{2x+1}$$

- Integrate. To calculate the integral on the left, substitute $u = y - 3$. Then $du = dy$. To calculate the integral on the right, substitute $v = 2x + 1$. Then $dv = 2dy$ and we have $dy = \frac{1}{2}dv$.
$$\int \frac{dy}{y-3} = \int \frac{dx}{2x+1}$$
$$\int \frac{du}{u} = \frac{1}{2}\int \frac{dv}{v}$$
$$\ln|u| + C_1 = \frac{1}{2}\ln|v| + C_2 \qquad \text{put the } x\text{'s and } y\text{'s back}$$
$$\ln|y-3| + C_1 = \frac{1}{2}\ln|2x+1| + C_2 \qquad \text{subtract } C_1 \text{ from both sides}$$
$$\ln|y-3| = \frac{1}{2}\ln|2x+1| + C_2 - C_1$$
Let $C = C_2 - C_1$. Then the equation becomes
$$\ln|y-3| = \frac{1}{2}\ln|2x+1| + C.$$

- Solve for C.
Substitute $x = 0$ and $y = 4$ and solve for C.
$$\ln|y-3| = \frac{1}{2}\ln|2x+1| + C$$
$$\ln|4-3| = \frac{1}{2}\ln|2(0)+1| + C$$
$$\ln(1) = \frac{1}{2}\ln(1) + C \qquad \text{the natural log of 1 is 0}$$
$$C = 0$$

- Solve for y.
$$\ln|y-3| = \frac{1}{2}\ln|2x+1| \qquad \text{use the law of logs that says } m\ln(a) = \ln(a^m)$$
$$\ln|y-3| = \ln|2x+1|^{1/2} \qquad \text{apply the exponential function to both sides}$$
$$e^{\ln|y-3|} = e^{\ln|2x+1|^{1/2}} \qquad e^x \text{ and } \ln(x) \text{ are inverses of each other}$$
$$|y-3| = |2x+1|^{1/2}.$$
This means that the general solution is either given by $y - 3 = |2x+1|^{1/2}$ or by $y = -|2x+1|^{1/2}$.