Homework Section 11.2 and 11.3.

I. Differential equations and tangent lines.

A. Look at Figure 24 on page 174. Which of the lines shown are tangent and which are not?

B. Recall that dy/dx is the slope of the tangent line to the curve y(x). Consider the Linear Differential Equation in problem #16 page 621.

$$\frac{dy}{dx} + 2y = e^{3x} \text{ and } y(0) = 50.$$

i. Find the slope of the tangent line to the graph of y(x) at the point (0,50) by evaluating the equation above at x=0 and y=50, and solving for dy/dx.

ii. Write the formula for the tangent line to the graph of y(x) at the point (0,50), using the point-slope formula (see page 6).

iii. Near the point (0,50), the tangent line at (0,50) is close to the graph of y(x). This means that the formula from part (B.ii.) is an approximation of the solution to the differential equation. Use the formula for the tangent line to approximate the solution to the differential equation at the point x = 0 + 0.1 = 0.1 (which is just a small distance from x=0).

C.

i. Write the formula for $\int e^{kx}\,dx$. (See page 375).

ii. Solve problem #16 page 621 (given above), using an integrating factor. Write your solution, clearly explaining each step as is done in Example 2 page 619.

iii. Evaluate your solution at x=0.1 to find the actual value that you approximated in (B.iii.). How close is your approximation to the true solution?

D.

i. Write the formula for $D_x(e^{kx})$. (Use the formula in the blue box at the bottom of page 235 to create this formula; see also Example 1 page 236.)

ii. Your solution to problem #16 page 621 is a formula for y(x). Calculate its derivative dy/dx. Evaluate the resulting dy/dx at the point (0,50) to find the slope of the tangent line at (0,50). Your answer should be the same as (B.i.).

II. These questions refer to the differential equation given in #6 page 628.

$$\frac{dy}{dx} = 1 + \frac{y}{x} \text{ and } y(1) = 0.$$

A.

i. Find the slope of the tangent line to the graph of the solution y(x) to the equation above at a point (x_0,y_0) by evaluating dy/dx at $x = x_0$ and $y = y_0$, (Your answer will be "slope = an expression with x_0 and y_0").

ii. Write the equation for the tangent line to the graph of y(x) at the point (x_0,y_0). Solve for y. Your answer will be "y= an expression with x, x_0 and y_0").

iii. Near the point (x_0,y_0), the tangent line at (x_0,y_0) is close to the graph of y(x). This means that the formula from part (B) is an approximation of the solution to the differential equation. Use the formula for the tangent line you found in (ii) to approximate the solution to the differential equation at the point $x_1 = x_0+h$ (which is a distance h from $x = x_0$).

B. A brief description of Euler's method for finding approximate values of a solution to a differential equation is given in the blue box on page 624. The result of Euler's method is not a formula for a solution, but a sequence of points (x_i, y_i) that lie close to the graph of the solution. See Figure 6 page 625; the red dots are the points near the graph of the solution, and the blue line is the true solution.
   i. The sequence of x-coordinates x_i of the points that approximate the graph of the solution are given by $x_{i+1} = x_i+h$. What is x_1 in terms of x_0? (Notice that $x = x_1$ is the value at which we approximated the solution in part (A.iii.).) If h=0.1 and x_0=0, write the first five terms of the sequence.
   ii. The sequence of y-coordinates y_i of the points that approximate the graph of the solution are given by $y_{i+1} = y_i+g(x_i,y_i)h$, where g(x,y) is the formula for dy/dx (in this problem, for example, g(x,y) = 1+y/x). Rewrite the formula for y_{i+1}, using g(x,y) = 1+y/x.
   iii. Use your answer to (ii) to find y_1 in terms of x_0, y_0 and h. Compare this result to your answer from part (A.iii.); they should be the same.
   iv. Using x_0=1 and h=0.1, find y_1, y_2 and y_3. (Note that in this problem, we are given y(1)=0, so we have y_0=0).

C. Solve problem #6 page 628 using an integrating factor, for x>0 (which tells us that |x|=x, so you can drop the absolute values when they turn up). Check yourself: the solution is y(x)=xln(x).

D. Comparing Euler's method to the actual solution.
- Launch Microsoft Excel.
- Type "x (h=0.1)" in cell A1, "Euler's Method" in cell B1, "Actual Solution" in cell C1 and "Error" in cell D1.

You will enter the value of x_0 into cell A2, x_1 into cell A3, x_2 into cell A4, etc…; similarly you will enter y_0 into cell B2, y_1 into cell B3, y_2 into cell B4, etc… Then, you will put the actual solution y(x) in column C, so that $y(x_0)$ is in cell C2, $y(x_1)$ is in cell C3, etc... In column D, you will compute the error.

- x_0=1, so enter 1 into cell A2. y_0=0, so enter 0 into cell B2.
- $x_1 = x_0+h$, and h=0.1, so enter the equation "=A2+0.1" into cell A3 (with the = but without the ""). Fill down to cell A12.
- $y_1 = y_0+(1+y_0/x_0)h$, and again h=0.1, so enter the equation "=B2+(1+B2/A2)*0.1" into cell B3. Fill down to cell B12.
- Using the actual solution you found in part C for y(x), we have y(x)=xln(x). In cell C2 we will put the Excel formula for $y(x_0)$=x0 $\ln(x_0)$. Since x_0 is in cell A2, you should type "=A2*ln(A2)" into cell C2. Fill down to cell C12.
- Put the difference between the Euler's method approximation and the actual solution in column D. In cell D2, type "=C2-B2". Fill down to cell D12.

You will now repeat these computations for h=0.05.
- Select the block of cells from A1 through D3. Choose copy from the Edit menu, or press control-c. Click on cell A15. Choose paste from the Edit menu.
- Change the label in cell A15 to "x (h=0.05)."

- Change the formula in cell A17 to "=A16+.05." Fill the formula in cell A17 down to cell A36. Similarly, replace the 0.1 by 0.05 in B17, and fill the formula in cell B17 down to cell B36, the formula in cell C17 down to C36, and the formula in D17 down to D36.

Now you will create a chart to visually represent the approximations and actual solution.

- Select cells A2 through C12.
- Choose Chart... from the Insert menu. A dialog box called "Chart Wizard – Step 1 of 4 – Chart Type" will appear. Click on x-y scatter plot. There are five chart subtypes shown. Click on the first one (with the dots only). Click Next.
- The title of the dialog box changes to Step 2 of 4 – Chart Source Data. Click the tab at the top of the dialog box that says Series.
    - Select Series 1. In the box labeled name, enter "Euler's Method (h=0.1)."
    - Select Series 2. In the box labeled name, enter "Actual Solution."
    - Click the "add" button. Series 3 appears in the list of series.
    - With Series 3 selected, enter "Euler's Method (h=0.05)" for the name. Click in the box entitled X-Values, and select cells A16 through A36. Click in the box entitled Y-Values, delete the contents of the box, and select cells B16 through B36.
    - Click Next.
- Step 3 of 4 – Chart Options.
    - Click on the Titles tab, if it is not already selected. Type "Comparing Euler's Method to the Actual Solution of a Differential Equation" for the title of your chart. For Value (X) axis, type "x," and for Value (Y) axis, type "y."
    - Click the tab at the top of the dialog box that says Gridlines. Check Major gridlines under Value (X) axis and Value (Y) axis.
    - Click Next.
- Step 4 of 4 – Chart Location. This dialog box allows you to choose whether the chart will appear in a new window, or as a part of the worksheet containing the data. Mark the "As an object in" radio button, if it is not already marked. Click Finish.
- Print out your chart and your worksheet separately and include it with your homework.

E. Read the paragraph following the blue box on page 624, and the paragraph following Figure 7 on page 627. Write a paragraph including the following:
  i. Explain how the error in column C is calculated.
  ii. As the values of x increase, is the error increasing or decreasing? Discuss both h=0.1 and h=0.05.
  iii. Explain that taking smaller values of h usually makes the models more accurate. Which model is more accurate, h=0.1 or h=0.05? Explain how you know.
  iv. What does the text claim is the preferred way to get greater accuracy?

Typing formulas into Excel. In the next problem, you will need to create formulas to type into Excel. When you want to type in multiplication, you must type a *.

For example if you want A2 times B2, you would type A2*B2; to type 3 times A2, you would write 3*A2. When you type the function e^x, you will type exp(x).

III. Solve the differential equation given in #3 page 628 analytically (without the spreadsheet), using techniques from either 11.1 or 11.2 (either will work here, since the equation is both separable and linear). Use your solution to find y(0.6). Then create a worksheet that shows Euler's approximation and the actual solution, and create a chart showing the Euler's Method approximation and the actual solution on the same axes. Here is some guidance to help you create your worksheet.

- The column headers (in row 1) will be the same as in (II.D.).
- Since the given point on the solution is y(0)=2, you will begin with $x_0=0$, so as in II.D., enter 0 in cell A2. You will use h=0.1, so enter =A2+0.1 in cell A3 and fill down until you reach the value 0.6, that the question asks about.
- Since the given point is y(0)=2, you have $y_0=2$, so you will enter 2 into cell B2. In cell B3, you will enter the formula for y_1. To get that formula, use the formula for y_{i+1} in the sequence given by Euler's Method, which says

  $$y_{i+1} = y_i + g(x_i, y_i)h$$

  (Don't know what to do? First, figure out what g(x,y) is in this problem, and substitute it into the equation above. Then put in i=0 to get the formula for y_1. To get the excel formula to type into cell B3, put in A2's everywhere that you have x_0, and B2 everywhere you have y_0. Use h=0.1.) Don't forget that when you type multiplication in Excel, you must type a *. Fill the formula you enter in B3 down to the same level you filled column A.

  Note, the answer to the question in the book will be found in column B at x=0.6. Check your answer in the back of the book.

- In cell C2, enter the formula for the solution that you found analytically, evaluated at x_0. This means that you should type your formula with x replaced by A2, since x_0 is in cell A2. Fill C2 down to the same level as you filled columns A and B.
- Create a chart showing both the actual solution and the Euler's method solutions, as you did for h=0.1 in part (II.D.).

Print your worksheet as well as your graph. Write your answer to the question in #3 page 628 on your worksheet. Say "The solution is approximately equal to ..... at x=0.6." How far off is the Euler Method approximation with h=0.1?