Calculus for Biology Newberger
Spring 2007

Homework Section 11.1.

Read the subsection of Chapter 11.1 entitled Logistic Growth, beginning on page 609.

I. Reading comprehension questions
  i. What is the equation for the standard model for unlimited growth?
  ii. What is the growth rate (dy/dx) proportional to in the standard model for unlimited growth?
  iii. What is the equation for the limited growth model? (This is discussed also in the paragraph above Example 5 on page 608).
  iv. What is the growth rate proportional to in the limited growth model?
  v. What is the equation for the logistic growth model?
  vi. What is the growth rate proportional to in the logistic growth model?
  vii. What is the solution for the logistic growth model, under the assumption that the population is positive, and less than the carrying capacity?
  viii. Copy the graph of the logistic curve for $b > 1$ into your homework. The solution to the logistic equation has two important properties: 1) the solution initially (meaning when time x is near 0) grows exponentially, and 2) the solution eventually (meaning when time x is very large), the population becomes very close to the carrying capacity. Label the part of your graph that looks exponential, and the part of the graph that shows the solution tending toward the carrying capacity.

II. You will solve #33 page 612, by doing the following steps.
  i. The logistic growth model is a separable differential equation. Rewrite it in the form $g(y)dy = f(x)dx$, i.e. with all the x's on the right hand side of the equation and all the y's on the left hand side of the equation. So that you can compare your work easily with the steps below, leave the k on the right hand side of the equation, so that it becomes part of $f(x)$. Say what $f(x)$ and $g(y)$ are.

  To integrate the left hand side of the equation, you must simplify this expression into an expression for which you know an antiderivative. The steps (ii-iv) will help you do that. Briefly explain what you are doing, as you answer these questions.

  ii. Rewrite $\left(1-\frac{y}{N}\right)y$ as a single fraction. Then invert it to rewrite $g(y)$.
  iii. In #33 part (a), you are told that $\frac{N}{(N-y)y} = \frac{1}{y} + \frac{1}{N-y}$. Verify this fact, by writing $\frac{1}{y} + \frac{1}{N-y}$ as a single fraction (find a common denominator and add). Use this fact to rewrite $g(y)$. (Tip: $(N-y)y$ is the same as $Ny-y^2$).
  iv. Rewrite your equation from (i), using your new expression for $g(y)$. Solve this equation by integrating both sides. (Tip: $\int \frac{1}{y} + \frac{1}{N-y} dy = \int \frac{1}{y} dy + \int \frac{1}{N-y} dy$; see the blue box at the bottom of page 384. Example 8 page 387 shows a similar integral, but be careful, because yours has a minus sign in it). Check yourself: your solution should be equivalent to $\ln|y| - \ln|N-y| = kx + C$.

v. In part (b) of #33, you are given the assumption that $0 < y < N$. Use this assumption to explain why you can eliminate the absolute values from your equation.

vi. Use the laws of logs (see page 91) to combine the left hand side of the equation into a single natural log. Say which law you used.

vii. Get rid of the natural log by applying the exponential function to both sides of your equation.

viii. Solve the equation for y. Check yourself: you solution should be equivalent to

$$y = \frac{e^{kx+C}N}{1+e^{kx+C}}.$$

Aside: If you are having trouble solving, here's a practice exercise. Do this on scratch paper; do not include it with your homework. Solve the equation $\frac{y}{a-y} = b$, using the following steps. First eliminate the fraction by multiplying on both sides by the denominator (a-y). Then distribute the b, so that $b(a\text{-}y)$ becomes $ba\text{-}by$. Now gather the y's to one side of the equation. Factor out the y, and use division to isolate the y, and get the solution. You should get $y = \frac{ba}{1+b}$.

ix. In part (viii), you found a formula for the population y, as a function of time x, as modeled by the logistic equation. We will now manipulate it into the form given on page 610 in equation (5). Begin with $y = \frac{e^{kx+C}N}{1+e^{kx+C}}$. We will multiply the right hand side of the equation by 1, which of course does not change the equation; the trick is that we will write 1 as $\frac{e^{-(kx+C)}}{e^{-(kx+C)}}$. Multiply the right hand side of the equation by $\frac{e^{-(kx+C)}}{e^{-(kx+C)}}$, and simplify both the numerator and the denominator. To simplify, use that $e^a e^{-a} = e^{a-a} = e^0 = 1$. Now use the distributive law and the laws of exponents to change $e^{-(kx+C)}$ into a product of e^{-C} and e^{-kx}. Your solution should now look similar to that on page 610. Say what b is in your equation. Note that since C is constant so is b. In the form of the solution given on page 610, the authors chose to write the constant as a single letter b instead of the more complicated constant that arises from integration.

x. In part (b) of #33, you are asked to show that $b = (N\text{-}y_0)/y_0$, where y_0 is the initial population size. Saying that y_0 is the initial population size is saying that, when time $x = 0$ (i.e. initially), the population $y = y_0$. Substitute these into your equation and solve for b to solve part (b) of #33.

III. A population is modeled by the logistic function

$$y = \frac{1000}{1+4e^{-2t}},$$

where t is time in days.

i. What is the initial population?

ii. How large can the population eventually get?

iii. After how long will the population reach 75% of the carrying capacity?