

In this exercise, you will study a population of insects using information about how they reproduce. These are short answer questions; so that your homework doesn't look like a list of random mathematics, either copy the questions or label your answers so that you can tell what they mean.

We begin with a population of insects with a lifespan of at most 2 years. We consider insects in their first year juveniles and insects between 1 and 2 years old adults. The population changes according to the following:

- 90% of the juveniles in a given year live to become adults in the next year.
- Both the juveniles and the adults reproduce, but at different rates. The number of juveniles in a given year is 60% of the number of juveniles the previous year plus 80% of the number of adults the previous year.

A. Let j_i be the number of juveniles in year i , and a_i be the number of adults in year i . (So for example a_0 is the number of adults in year 0.)

- If you have 100 juveniles and 200 adults in year 2 (i.e. if $j_2 = 100$, and $a_2 = 200$), how many juveniles and adults will there be in year 3 (i.e. what are j_3 and a_3)?
- Write formulas for j_{i+1} and a_{i+1} in terms of a_i and j_i . Then use a matrix to write your formulas so the result is in the form

$$\begin{bmatrix} j_{i+1} \\ a_{i+1} \end{bmatrix} = \begin{bmatrix} p & q \\ s & t \end{bmatrix} \begin{bmatrix} j_i \\ a_i \end{bmatrix},$$

where p , q , s and t are numbers.

- In the Excel part of this assignment, you will enter a number for j_0 in cell D2, a_0 in cell E2, j_1 in cell D3, a_1 in cell E3, and so forth. You will enter your

Leslie matrix so that $\begin{bmatrix} A2 & B2 \\ A3 & B3 \end{bmatrix} = \begin{bmatrix} p & q \\ s & t \end{bmatrix}$. So we will have

$$\begin{bmatrix} D3 \\ E3 \end{bmatrix} = \begin{bmatrix} A2 & B2 \\ A3 & B3 \end{bmatrix} \begin{bmatrix} D2 \\ E2 \end{bmatrix}.$$

Multiply the matrices on the right to find formulas for D3 and E3 in terms of cells A2, A3, B2, B3, D2 and E2.

B. This part is about what you can say when your data fall on a line through the origin.

- You will use Excel to find the ratio of adults to juveniles in the population each year, and discover that that ratio becomes constant from year to year. Suppose that the ratio of adults to juveniles is m , i.e. $a/j = m$ for all of the points (j, a) in the data after a certain year. Write a as a function of j ; express your answer like this: write a sentence that says that if the ratio of adults to juveniles is m , then $a =$ your answer. Write a sentence to explain that this relationship between a and j means that the graph of the data is a line, and say what the slope and y -intercept of that line are.
- Suppose that the ratio of adults to juveniles eventually becomes 0.75. Then from the above, you should see that the data eventually lie on a line; what is the slope

and y-intercept of that line? Give an example of a population (saying how many juveniles and adults there are) that lies on that line.

Follow the Excel instructions below before answering these questions.

- C. Now you can change the values in cells D2 and E2 to see what happens to the populations with different initial conditions.
- Enter 75 in cell D2 and 25 in cell E2. Note that the numbers in columns D, E, F, G and H change automatically, as do the graphs. Enter various populations in cells D2 and E2, to see how the data and graph of adult population versus juvenile population vary. Write a sentence explaining what happens to the ratio of adults to juveniles as you change the initial conditions.
 - Notice that the graph of the adult population versus the juvenile population does not generally look like a line in the first few years. Find an initial population (number of juveniles and number of adults) so that the graph of the data forms a line from year 0. Print the graph of the data set for the initial condition you found. (Hint: You know what the ratio of adults to juveniles should be for that population.)
- D. Now consider the graph of the total population. This function appears exponential, by its shape, so we used an exponential function to model it. Suppose that a population changed by a factor k from year to year. Say the population in year i is p_i . Then saying the population changes by a factor of k each year means, $p_{i+1} = kp_i$.
- Look in column H. By what factor does the population *eventually* change from year to year? (i.e. what is k in this case?)
 - To see how populations with this behavior evolve, do the following computation.
 - Write p_1 in terms of p_0 .
 - Write p_2 in terms of p_1 , and then substitute from part (a) to write p_2 in terms of p_0 .
 - Write p_3, p_4 , and p_5 in terms of p_0 .
 - Use the pattern you see in parts (a) through (c) to write a general formula for p_i in terms of p_0 .
 - Now let us consider the function $f(x)$ that yields the population after x years. Then $f(0) = p_0, f(1) = p_1$, etc...
 - While the numbers p_i only make sense when i is a whole number of years, we will use the relationship you found in part (ii.d.) to model the population after any real number of years (like half a year, for example), simply by allowing the number we plug in for time to be a real number. Write the formula for $f(x)$ by replacing the i in the formula from part (ii.d.) by x .
 - Now $f(x)$ is a constant times an exponential function (see page 27). In this exercise, we will rewrite this function using the natural exponential function e^x (e is a convenient base because the derivative of e^x is particularly easy). We have k^x in our formula. We want to find a number r so that

$$k^x = e^{rx}.$$
 Solve the above equation for r (showing your work). (See Example 5 part (a) page 95 for help with solving exponential equations.)

- c. Now put the value of k that you found in part (i), into your formula for r , to get a numerical value for r . Compare your result with the number in the exponent of the model (what Excel calls the trendline) that you fit to the graph of the total population.

Excel Instructions

Using the Leslie matrix you created in part A, you will use Microsoft Excel to create data (j_i, a_i) for years $i = 0, 1, 2, 3, \dots$, and you will find by plotting the data that after a while, the points (j_i, a_i) eventually lie in a line. You will calculate the ratio a_i/j_i to determine the slope of this line, which represents the long term behavior of the population.

- Launch Microsoft Excel.

Setting up the Leslie Matrix

- Type “Leslie matrix” in cell A1, “year” in cell C1, “juveniles” in cell D1 and “adults” in cell E1. Adjust the column widths as necessary. The Leslie Matrix will be in columns A and B.
- Enter the following Leslie matrix in the given cells.

$$\begin{bmatrix} A2 & B2 \\ A3 & B3 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ 0.9 & 0 \end{bmatrix}$$

- In cell C2 type the number 0. In cell C3, type the number 1. Select both cells C2 and C3, and put your cursor on the bottom right hand side of the cell. Your cursor should change into a small black plus shape. Hold the mouse so that the cursor shows that plus shape, press the left mouse button, and holding it down, drag to cell C23. The numbers 0 through 21 should now appear in column C.

Calculating the population in the first 21 years

- We will first consider a population that has 100 juveniles and no adults in year zero. Enter 100 in cell D2 and 0 in cell E2.
- Now you will use the formulas you found in part (I.A.iii.) to put the number of juveniles in year 1 into cell D3, and the number of adults in year 1 into cell E3. Enter $=A\$2*D2+B\$2*E2$ into cell D3, and

$$=A\$3*D2+B\$3*E2$$

into cell E3 (including the equals signs). When we fill these formulas down, the D2 and E2 will change to D3 and E3, but the A2, A3, B2 and B3 will remain unchanged, because of the dollar signs.

- Fill the formulas you entered in Cells D3 and E3 to row 23.

Plotting the data with adults in the y -axis and juveniles in the x -axis for the first 10 years

- Select cells D2 through E12. Note that this is only part of your data.
- Choose Chart... from the Insert menu. A dialog box called “Chart Wizard – Step 1 of 4 – Chart Type” will appear. Click on x-y scatter plot. There are five chart subtypes shown. Click on the first one (with the dots only). Click Next.

- The title of the dialog box changes to Step 2 of 4 – Chart Source Data. We won't use these options at this time. Click Next.
- Step 3 of 4 – Chart Options.
 - Click on the Titles tab, if it is not already selected. Type “Adults vs. Juveniles” for the title of your chart. For Value (X) axis, type “Juvenile population,” and for Value (Y) axis, type “Adult Population”
 - Click the tab at the top of the dialog box that says Gridlines. Check Major gridlines under Value (X) axis and Value (Y) axis.
 - Click the tab at the top of the dialog box that says Legend. Uncheck the box that says Show legend.
 - Click Next.
- Step 4 of 4 – Chart Location. This dialog box allows you to choose whether the chart will appear in a new window, or as a part of the worksheet containing the data. Mark the “As an object in” radio button, if it is not already marked. Click Finish.
- Click on the white background of the chart to select it, and choose Print from the File menu. Since the chart was selected when you chose Print, Excel will print just the chart.
- So that you can see how the population changed over time, label each data point on your printout with the corresponding year. Tip: when you hold your mouse over a data point on your scatter plot, Excel will tell you the coordinates of that point. Look at your worksheet to find the year.

The data that you have just plotted becomes linear as time goes on. You will calculate the slope of this line by calculating the ratio of adults to juveniles in each year. Since the data becomes linear, this ratio will become constant.

- Type “Ratio of Adults to Juveniles” in cell F1. In cell F2, type “=E2/D2.” Fill that equation down to cell F23.

Let us now consider the growth of the total population from year to year.

- Type “Total Population” in cell G1. In cell G2, type “=D2+E2.” Fill that equation down to cell G23.
- Type “Growth” in cell H1. In cell H2, type “=G3/G2.” Fill that equation down to cell H23. This makes the total population in year 3 (in cell G5), for example, H4 times the population in year 2 (in cell G4).

Plotting the total population over time

- Select cells G2 through G23.
- Choose Chart... from the Insert menu. A dialog box called “Chart Wizard – Step 1 of 4 – Chart Type” will appear. Click on x-y scatter plot. There are five chart subtypes shown. Click on the first one (with the dots only). Click Next.
- The title of the dialog box changes to Step 2 of 4 – Chart Source Data. Click the tab at the top of the dialog box that says Series. Click in the window labeled x-values.

On your spread sheet, select cells C2 through C23. The code =Sheet1!\$C\$2:\$C\$23 should appear in the window labeled x-values. Click Next.

- Step 3 of 4 – Chart Options.
 - Click on the Titles tab, if it is not already selected. Type “Total Population over Time” for the title of your chart. For Value (X) axis, type “Time (years),” and for Value (Y) axis, type “Total Population”
 - Click the tab at the top of the dialog box that says Gridlines. Check Major gridlines under Value (X) axis and Value (Y) axis.
 - Click the tab at the top of the dialog box that says Legend. Uncheck the box that says Show legend.
 - Click Next.
- Step 4 of 4 – Chart Location. This dialog box allows you to choose whether the chart will appear in a new window, or as a part of the worksheet containing the data. Mark the “As an object in” radio button, if it is not already marked. Click Finish.

Fitting an exponential model to this data

- Right click on one of the data points in the graph of the total population versus time, and select Add trendline... from the pop-up menu that appears. A dialog box entitled Add Trendline will appear.
- Click on the tab labeled Type at the top of the dialog box, if it is not already selected. Click Exponential.
- Click on the tab labeled Options at the top of the dialog box. Mark the checkbox labeled Display Equation on Chart.
- Click OK.
- The graph of an exponential function and its formula will appear on the chart with your data.
- Click on the white background of the chart to select it, and choose Print from the File menu. Since the chart was selected when you chose Print, Excel will print just the chart.