Calculus for Biology Newberger

Spring 2007

Final Exam Review Sheet

This exam covers parts of Chapters 10, 11, 12 and 13, plus part of the packet on Differential Equations and Difference Equations, as described below.

Matrices
1 page for material from Section 10.1
1 page for material from Section 10.5

Differential Equations
1 page for material from Sections 11.1 and 11.2
1 page for material from Section 11.4

Probability and Continuous Random Variables
1 page for material from Section 12.2 and 12.3
1 page for material from Sections 13.1 and 13.2

Section 10.1 (covered on Exam 2)
You *will* be asked to

- Solve a system of equations using Gauss-Jordan elimination. There are three possible results: the system will either have a unique solution, no solution or infinitely many solutions. In the case that a system has infinitely many solutions, you will be asked to write the general solution for that system, and you may be asked to give an example of numbers that satisfy the system. When the solution does not exist, you will be asked to explain how you can tell from the augmented matrix that this is the case.

Note that you use the techniques from 10.1 in answering two other types of problems covered on this exam. They are

- From 10.5, Example 2 page 592 part (a): Given vectors X(0), V_1 and V_2, find numbers a and b such that $X(0)=aV_1+bV_2$. This is discussed in detail below, in the discussion of material from Section 10.5 from after Exam 3.
- After you have the eigenvalues, you use row reduction techniques to calculate the eigenvectors. When you are calculating the eigenvectors, you will always get an infinite number of solutions, and express the eigenvectors in general form.

Section 10.5 (covered on Exam 2)
You *will* be asked to

- Calculate the eigenvalues and eigenvectors for a given matrix. This may be part of solving a problem involving a Leslie matrix for a population, or part of finding a solution for a system of differential equations from 11.4. See below.
- When there are an infinite number of solutions to a system of equations, the general solution to a system of equations has a variable that could be any real number. This is the case when finding eigenvectors; there are always an infinite number of eigenvectors corresponding to each eigenvalue. You may be asked to give a particular example of an eigenvector. Find this by choosing a number for the variable in the general solution. For example, if your eigenvectors for a particular eigenvalue

are of the form $(3y, y)$, then taking $y = 1$ gives the example (3,1). Remember that eigenvectors are nonzero vectors, so taking $y= 0$ will not give you an example of an eigenvector.

You may be asked to

- State the definition of an eigenvector. A *nonzero* vector v is an eigenvector for a matrix A if $Av = \lambda v$ for some real number λ.
- Calculate an eigenvector given an eigenvalue.
- Verify that a given vector is an eigenvector and determine its eigenvalue. You did this in problem 5e on Exam 2.

Section 10.5 (covered after Exam 3)
You will be asked about a Leslie Matrix model for the evolution of a population.

- You may be asked to calculate the numbers of juveniles and adults in a particular year, given the Leslie Matrix and the population in the previous year. For example, if you are given that the Leslie Matrix is $M = \begin{bmatrix} 0.6 & 0.8 \\ 0.9 & 0 \end{bmatrix}$ and that there are 100 juveniles and 400 adults in year 7, you can calculate the number of juveniles and adults in year 8 by matrix multiplication:
$$M\begin{bmatrix} 100 \\ 400 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ 0.9 & 0 \end{bmatrix}\begin{bmatrix} 100 \\ 400 \end{bmatrix} = \begin{bmatrix} 380 \\ 90 \end{bmatrix}.$$
Thus in year 8, there will be 380 juveniles and 90 adults.

- You may be given the Leslie Matrix and be asked to give a formula for the numbers of juveniles and adults in an arbitrary year (like year n), in terms of the population in the previous year. For example, if there are jn-1 juveniles and an-1 adults in year n-1, then you can give a formula for the populations in year n by calculating:
$$\begin{bmatrix} j_n \\ a_n \end{bmatrix} = M\begin{bmatrix} j_{n-1} \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ 0.9 & 0 \end{bmatrix}\begin{bmatrix} j_{n-1} \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 0.6j_{n-1} + 0.8a_{n-1} \\ 0.9j_{n-1} \end{bmatrix}$$
So the number of juveniles in year n is $j_n = 0.6\, j_{n-1}$+0,8 j_{n-1}, and the number of adults in year n is $a_n = 0.9\, j_{n-1}$.

- You will be given a Leslie Matrix and asked (a) to find a population (number of juveniles and number of adults) of size 10,000 for which the proportion of the population in each age group stays the same from one year to the next, (b) tell by what factor the population grows or declines each year. See problems 14-19 page 594.
    - Note that to answer these questions, you will use the eigenvalues and corresponding eigenvectors. To answer this question, you need only consider positive eigenvalues.
    - Because of time constraints, you may be given the eigenvales and eigenvectors for the matrix.

- You may be given a Leslie Matrix and asked to find a formula for the population in year *n*. You do this using the techniques explained in Example 2 page 592. See also problems 20 and 21 page 594.
  - To answer this question, you will use both of the eigenvalues for the matrix, no matter if they are positive or negative.
  - As in part (a) of Example 2 page 592, you may be asked a question like the following (using 10.1 techniques).
    - Given vectors X(0), V_1 and V_2, find numbers a and b such that X(0)=aV_1+bV_2.
    - For example, given vectors $X(0)=\begin{bmatrix}4\\2\end{bmatrix}$, $V_1=\begin{bmatrix}1\\1\end{bmatrix}$ and $V_2=\begin{bmatrix}0\\3\end{bmatrix}$, find numbers a and b such that X(0)=aV_1+bV_2. To find *a* and *b*, solve the system $\begin{bmatrix}1&0\\1&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$, using the techniques of Section 10.1. The answer is $a = 4$, and $b = -2/3$. Check your answer by checking that $4\begin{bmatrix}1\\1\end{bmatrix}-2/3\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$.
  - As in part (c) of Example 2 page 592, you will be asked to find a formula for the long-term population distribution. Your answer will be of the form:

$$X(n)=\begin{bmatrix}j_n\\a_n\end{bmatrix}=(number)(eigenvalue\#1)^n\left[eigenvector\#1\right]+(number)(eigenvalue\#2)^n\left[eigenvector\#2\right]$$

  - Also as in part (c) of Example 2 page 592, you may be asked to find an approximate expression for the long-term population distribution. To answer this, argue that if an eigenvalue λ is less than 1, then when *n* is large, λ^n is very close to zero. Use this argument to approximate the long-term population distribution.

Differential Equations
Section 11.1 and 11.2 (covered on Exam 3)
Vocabulary
You may be asked what form is taken by differential equations of the following types. These questions may be short answer or fill in the blank.

- Unlimited growth model (page 608), including what the constant *k* stands for, and what the variables *x* and *y* represent.
- Limited growth model (page 609), including what the constants *k* and *N* stand for, and what the variables *x* and *y* represent.
- Logistic growth model (page 609), including what the constants *k* and *N* stand for, and what the variables *x* and *y* represent.

You *will* be asked at least one of the following:

- Find a general or particular solution to a separable differential equation (11.1).
- Find a general or particular solution to a linear first order differential equation (11.2).

Section 11.4 (covered after Exam 3)
You will be asked to solve a system of linear differential equations.

- You will be given a system of differential equations (see for example problems 1-8 page 637). You may be asked to write the system as a matrix equation of the form $\frac{dX}{dt} = MX + Q$. For example the system

$$\frac{dx_1}{dt} = 4x_1 - 2x_2 + e^t$$
$$\frac{dx_2}{dt} = -3x_1 + 9x_2 + e^{2t}$$

  becomes

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}$$

  in matrix form.

- The page dedicated to this section may be where you are asked to find the eigenvalues and eigenvectors for a matrix. That way the page on 10.5 can be dedicated to understanding the Leslie Matrix model for population. You will not be asked to calculate all the eigenvalues and eigenvectors for more than one matrix, in any case.

Probability (covered on Exam 1)
The final exam *will* include a page concerning specificity and sensitivity of a medical test, not unlike that problem on Exam 1.

Probability
Section 12.1
You *may* be asked to
- Use given information to write the numbers of elements or probabilities in each region of a Venn diagram. **Practice:** Example 10 page 666, #58 page 671, #73-75 page 673, #66 page 688, Example 2 page 694.
- Label the regions of a Venn diagram with its name (for example $A \cap B \cap C'$). **Practice:** Hand graded homework 12.1 (I.A.i.), #31, 33, 38,41 page 670 (these are different, but related).
- Read information given in set notation off of a table. **Practice:** Example 12 page 668, #59-66 page 671-672.

Section 12.3
You *will* be asked to
- Work with the sensitivity and specificity of a medical test.
- Calculate conditional probabilities (the probability that something happens given that something else happened). **Practice:** Examples 2 and 3 page 694, #1-6 page 703.
- Use the product rule for probabilities. **Practice:** #51 page 707.

You *may* be asked to
- Calculate conditional probabilities from a table. **Practice:** Example 1 page 693, #38 page 705.

Continuous random variables
Section 13.1
You *will* be asked to
- Calculate probabilities using a probability density function. Practice: Example 3 page 733, #27, 33 page 736, see also #11-14 page 744, parts d and e.

You *may* be asked to
- Shade areas on the graph of a probability density function representing given probabilities. See Figure 2 page 730, Example 3 page 753.
- Determine whether or not a given function is a probability density function on a given integral. **Practice:** Example 1 page 731, #3-8 page 735.

Section 13.2
You *will* be asked to
- Calculate the mean (expected value) and standard deviation of a continuous random variable, given a formula for its probability density. **Practice:** Example 1, 2, 3 pages 740-742, #11-14 page 744, #21, 23, 25 page 745. Also, #1-8 page 744.

You *may* be asked to

- Determine the probability that a random variable is within a given number of standard deviations of its mean. **Practice:** #11-14 page 744 part e.

Integration:
The problems from 11.1, 11.2, 13.1 and 13.2 involve integration. You *may* be asked to solve integrals using the following techniques and formulas.

- Power rule for integration (page 371).
- Integral of x^{-1} (page 374).
- Integral of e^{kx} (page 375).
- Substitution, in particular to solve integrals similar to the following.

$$\int xe^{x^2}\,dx \text{ or } \int \frac{t}{t+1}\,dx$$

Note that both of these came up in WeBWork problems. Substitution is covered in Section 7.2.