Newberger MATH 550A Spring 05

1. a. (6 points) State the definition of the topology generated by a metric. You do not need to define a metric; simply define the topology in a metric space.
   b. (10 points) Prove that a metric space is a Hausdorff topological space. If you state $A \cap B = \emptyset$ for some sets A and B, then prove it.

   Let (X, d) be a metric space, and let $x, y \in X$ be distinct. Let $r = d(x, y)/2$. Then $B(x, r)$ and $B(y, r)$ are open neighborhoods of x and y respectively. Furthermore, we can show $B(x, r) \cap B(y, r) = \emptyset$. Suppose that $z \in B(x, r) \cap B(y, r)$. Then

   $$d(x, y) \leq d(x, z) + d(z, y) < r + r = d(x, y),$$

   which is a contradiction. Thus $B(x, r) \cap B(y, r) = \emptyset$. It follows that X is Hausdorff.

2. a. (6 points) State the definition of a connected topological space.
   b. (10 points) Prove that the image of a connected space under a continuous function is connected.
   c. (14 points: 10 for one, 14 for both) Consider the set $Y = [0, 1] \cap \mathbb{Q}$ as a subspace of $\mathbb{R}$ with the standard topology. Answer the following questions, giving a proof that your statement is correct.
      i. Is Y connected?

      Y is not connected since the set $(-1, \sqrt{2}/2) \cap Y$ and $(\sqrt{2}/2, 2) \cap Y$ form a separation of Y.

      ii. Is Y compact?

      Y is not compact, since Y is not closed. Any irrational number in [0,1] is a limit point for Y, and yet is not contained in Y, hence Y is not closed.

3. a. (6 points) State the definition of a compact topological space.
   b. (10 points) Let X be a topological space. Prove that the union of finitely many compact subspaces of X is compact.
   c. (10 points) Is the union of countably many compact subspaces of X compact? Prove it is or give a counterexample.

   The union of countably many compact subspaces may not be compact. For example, the one point sets $\{n\}$, where n is an integer, are each compact under the subspace topology induced by the standard topology on $\mathbb{R}$, but their union, $\mathbb{Z}$ is not compact, since it is infinite and caries the discrete topology.

4. Consider the following topology on $\mathbb{R}$:

$$\mathcal{T}_0 = \{A \subset \mathbb{R} | 0 \in A \text{ or } A \text{ is empty}\}.$$

a. (7 points) Let B be a subset of $\mathbb{R}$ that contains 0. Prove that B is connected with respect to $\mathcal{T}_0$.

Any separation of B would consist of two disjoint nonempty open sets. However, since $0 \in B$, open subsets in B either contain 0 or are empty. Thus all nonempty open sets intersect at 0, and are not disjoint. Thus B is connected.

b. (7 points) Give an example of a subset of $\mathbb{R}$ that is not connected with respect to $\mathcal{T}_0$.

If X is a subset of $\mathbb{R}$ that does not contain 0, then the supbspace topology induced on X is the discrete topology. Here's why: if $x \in X$ then the set $\{x, 0\}$ is open in $\mathcal{T}_0$. Thus since $\{x\} = \{x, 0\} \cap X$, the one point set $\{x\}$ is open in X. Thus X has the discrete topology. Now take for example the set $\{1, 2\}$. This set is not connected since $\{1\}$ and $\{2\}$ form a separation.

c. (7 points) Let C be an infinite subset of $\mathbb{R}$. Prove that C is not compact with respect to $\mathcal{T}_0$.

Consider the open cover of C given by $\{\{x, 0\} | x \in C\}$. This does not have a finite subcover, since any finite subcollection would only cover a finite number of points.

d. (7 points) Prove that $(\mathbb{R}, \mathcal{T}_0)$ is not metrizable. (Hint: Prove $(\mathbb{R}, \mathcal{T}_0)$ is not Hausdorff).

There is no open set containing the number 2 that does not contain 0, so 2 and 0 cannot be separated by open sets.