Topology I, Newberger, Spring 2005

Homework: Sections 19 and 30

Due April 21

I. (20 points)
   A. Read the definition of a subbasis on page 82.
   B. Read on page 114, starting with "Now we generalize...," and ending with Theorem 19.1. The following parts refer to notation from these pages. Note that $\mathbb{R}^\omega$ is the product $\prod_{i=1}^{\infty} X_i$, where $X_i = \mathbb{R}$ for all $i = 1, 2, \ldots$.
   C. This part is about the projection maps.
      i. Consider the product space $\mathbb{R}^3$. What is J in this case? Describe the projection functions π_β, in the particular case of the product space $\mathbb{R}^3$. Draw a diagram that illustrates the definition of these maps.
      ii. Consider the product space $\mathbb{R}^\omega$. What is J in this case? Describe the projection functions π_β, in the particular case of the product space $\mathbb{R}^\omega$. Give an example of π_β applied to a particular point of $\mathbb{R}^\omega$ that illustrates the definition of this map.
      iii. Consider the general product space $X = \prod_{\beta \in J} X_\beta$. For each $\beta \in J$, what are the domain and codomain of the projection map π_β? Is π_β one-to-one? Is π_β onto? Explain (briefly).
   D. This part is about the subbasis for the product topology.
      i. For the general product space $X = \prod_{\beta \in J} X_\beta$, write the definition of the sets $\mathcal{S}_\beta$, for $\beta \in J$, and $\mathcal{S}$. Where does $\mathcal{S}_\beta$ live?
      ii. In the case of $\mathbb{R}^\omega$, give an example of an element of $\mathcal{S}_\beta$ that illustrates the nature of the sets in $\mathcal{S}_\beta$.
      iii. Verify that $\mathcal{S}$ is a subbasis for a topology, by proving $\prod_{\beta \in J} X_\beta = \bigcup_{S \in \mathcal{S}} S$.
   E. This part is about the basis for the product topology as it is defined in these pages.
      i. Write the definition of the basis $\mathcal{B}$ generated by a subbasis $\mathcal{S}$.
      ii. In the case of the product space $\mathbb{R}^3$, give examples of elements of $\mathcal{B}$ that illustrate the nature of this basis. In addition to any verbal description you deem necessary, draw diagrams showing your examples, and write your examples down in terms of the various projection maps.
      iii. In the case of the product space $\mathbb{R}^\omega$, give examples of typical basis elements in $\mathcal{B}$ for the product topology.

II. (10 points) Prove that the product topology on $\mathbb{R}^\omega$ is second countable.

III. (10 points) Do problem #4 page 194.