Topology I, Newberger, Spring 2005

Tips and Remarks: Sections 20 and 21.

I.B. Fundamental: Prove the metric in #1.a. induces the product topology on $\mathbb{R}^n$.

We showed that the square metric ρ given on page 122 induces the product topology. Use this in problem 1.a. Show that the square metric ρ and the given metric d' induce the same topology. To do this, first show that

$$\rho(x, y) \leq d'(x, y) \leq n\rho(x, y).$$

Then use Lemma 20.2 to prove that these metrics induce the same topology. (Use the Lemma twice: once to prove that the topology generated by ρ is finer than the one generated by d', and the second time to prove the topology generated by d' is finer than the one generated by ρ.) Use the proof of Theorem 20.3 as a model.

III.B. Prove that if (x_n) converges to x and (y_n) converges to y then $x_n \times y_n$ converges to $x \times y$ in the product topology on $X \times Y$.

Start with what you want to prove. You want to show $x_n \times y_n \to x \times y$. I.e. you want to show that for every neighborhood V of $x \times y$, there exists N such that $n \geq N$ implies that $x_n \times y_n \in V$. This is a for every statement. Start with let. Let V be a neighborhood of $x \times y$ in $X \times Y$.

Since V is open in $X \times Y$, and $x \times y \in V$, there exists a basis element for the product topology $A \times B$, where A is open in X and B is open in Y, such that $x \times y \in A \times B \subset V$.

Now you have $x_n \to x$. This means for every neighborhood U of x there is an N such that $n \geq N$ implies that $x_n \in U$. The only way to use a "for every set U" statement like this one is to specify which set U you are applying it to. If you like to write the definition of $x_n \to x$ in your argument, try it this way:

Since $x_n \to x$, we have by definition that for every neighborhood U of x there is an N such that $n \geq N$ implies that $x_n \in U$. In particular, there exists N such that $n \geq N$ implies that $x_n \in A$ (applying the definition to $U = A$, where A is defined above).
Do something similar for B. Complete the proof.

IV. Do problem #8 on page 134.

1. Write out what you want to show. Let $\varepsilon > 0$
2. Use the triangle inequality to get

$$d(f(x), f_n(x_n)) \leq d(f(x), f(x_n)) + d(f(x_n), f_n(x_n)).$$

3. Prove (applying the Uniform Limit Theorem 21.6) that f is continuous.
4. Prove (applying Theorem 21.3) that the sequence of points $(f(x_n)) \subset Y$ converges to $f(x)$. Use this to control the size of $d(f(x), f(x_n))$ in terms of ε.
5. Use uniform convergence to control the size of $d(f_n(z), f(z))$ in terms of ε for all z. Since it is for all z, you can apply it to $z = x_n$ and you will have control over $d(f_n(x_n), f(x_n))$.
6. Complete the proof, showing that there exists N such that $n \geq N$ implies $d(f(x), f_n(x_n)) < \varepsilon$.