

Topology I, Newberger, Spring 2005

Homework: Sections 17 and 18: Tips and Remarks

IIB#11 page 101. Let X and Y be Hausdorff topological spaces. The key tip for this problem is to set up the “what you want” part correctly.

Write what you want: We want to prove that $X \times Y$ is Hausdorff.

Write what this means: This means that we want to show the following for every statement:

For every $(x_1 \times y_1), (x_2 \times y_2) \in X \times Y$ with $(x_1 \times y_1) \neq (x_2 \times y_2)$, there exist open sets W_1 and W_2 in $X \times Y$ with $(x_1 \times y_1) \in W_1$ and $(x_2 \times y_2) \in W_2$, and $W_1 \cap W_2 = \emptyset$.

To prove a for every statement, you need some set up: Let $(x_1 \times y_1), (x_2 \times y_2) \in X \times Y$ with $(x_1 \times y_1) \neq (x_2 \times y_2)$.

Now restate what you want for these particular points $(x_1 \times y_1)$ and $(x_2 \times y_2)$: We want to find open sets W_1 and W_2 in $X \times Y$ with $(x_1 \times y_1) \in W_1$ and $(x_2 \times y_2) \in W_2$, and $W_1 \cap W_2 = \emptyset$.

Now you have $(x_1 \times y_1) \neq (x_2 \times y_2)$.

This means that $x_1 \neq x_2$ or $y_1 \neq y_2$ (or both).

Case 1. Suppose $x_1 \neq x_2$. Now you can use your assumption that X is Hausdorff, applied to these two particular points x_1 and x_2 of X , to find U_1 and U_2 that are open in X , disjoint, and contain x_1 and x_2 respectively. Prove that $W_1 = U_1 \times Y$ and $W_2 = U_2 \times Y$ satisfy the requirements for the sets you wanted to find. In other words, prove that these two sets are open (this is easy since all basis elements are open, even though not all open sets are basis elements) and that they are disjoint.

Now do Case 2 $y_1 \neq y_2$. (You can write that the proof is similar, or write it out to reinforce the previous part, as you see fit.)

Your assumptions in that problem were that X and Y are Hausdorff. Since X is Hausdorff, we know for every $x_1, x_2 \in X$, there exists U_1, U_2 open in X such that $x_1 \in U_1, x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. This for every statement is in the assumptions. To use a statement “for every $x_1, x_2 \in X$ ” in your proof, you must apply it to a particular $x_1, x_2 \in X$. In our proof above, we started with particular points $(x_1 \times y_1), (x_2 \times y_2) \in X \times Y$, and used them to come up with particular elements of X to which we applied our for every statement.

IIIBii.#11 page 112. Let $F : X \times Y \rightarrow Z$, and let $y_o \in Y$. Let $h : X \rightarrow Z$ be given by $h(x) = F(x, y_o)$. We will suppose F is continuous, and prove that h is continuous. So suppose that F is continuous. Once again begin the argument by writing what you want to prove. Writing what the assumptions mean is not going to get you to the end! Start with what you want.

There are several equivalent definitions of continuity. We could prove

- (1) For every open set $V \subset Z$, $h^{-1}V$ is open in X ; or
- (2) For every $x \in X$ and every neighborhood V of $h(x)$ in Z , there exists a neighborhood U of x such that $h(U) \subset V$; or
- (3) etc...

I suggest you use (2). It will reinforce the who “for every” business, and it leads to a nice proof.

So we want to show for every $x \in X$ and every neighborhood V of $h(x)$ in Z , there exists a neighborhood U of x such that $h(U) \subset V$.

This is a for every statement, so start with let...

Let $x \in X$ and let V be a neighborhood $h(x)$ in Z .

Show $x \times y_o \in F^{-1}V$.

Since F is continuous, $F^{-1}V$ is open in $X \times Y$. This does *not* mean that $F^{-1}V = A \times B$ where A is open in X and B is open in Y . $F^{-1}V$ is an open set, but it is not necessarily a basis element. The basis for the topology on $X \times Y$ is all the sets of the form $A \times B$ where A is open in X and B is open in Y . The open sets are unions of basis elements, but they may not actually be basis elements.

Even though you cannot say that $F^{-1}V = A \times B$, you do know that there is a basis element around each point in $F^{-1}V$ that is contained entirely in $F^{-1}V$. Use this to explain that there are open sets $A \subset X$ and $B \subset Y$ such that $x \times y_o \in A \times B \subset F^{-1}(V)$. A is the open set (U in the statement) that you are looking for. To finish the argument, show that A satisfies the requirements: show that A is a neighborhood of x with $h(A) \subset V$.

IIIC #12(c) page 112 In part (b) of this problem, you showed that

$$g(X) = F(x, x) = \begin{cases} 1/2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

To prove that F is not continuous, we will prove that there is a closed set C in \mathbb{R} such that $F^{-1}C$ is not closed in $\mathbb{R} \times \mathbb{R}$.

To show that $F^{-1}C$ is not closed, we will show that there is a limit point of $F^{-1}C$ that is not contained in $F^{-1}C$. (This proves $F^{-1}C$ is closed, right? Closed sets contain all of their limit points.)

Let $C = \{1/2\}$. Then by your work in part (b), you have that

$$F^{-1}C \supset \{x \times x | x \in \mathbb{R} \text{ and } x \neq 0\}$$

Show that

- (1) $0 \times 0 \notin F^{-1}C$.
- (2) 0×0 is a limit point of $F^{-1}C$.

To do (2), show that any open set containing 0×0 intersects $F^{-1}C$ at a point different from 0×0 . (This is the definition of limit point.) “Any open set...” This is a for every statement in disguise! Prove that *every* open set in $\mathbb{R} \times \mathbb{R}$ that contains 0×0 intersects $F^{-1}C$. Start with let... Let W be an open set in $\mathbb{R} \times \mathbb{R}$ that contains 0×0 . Then (even though W might not have the form $A \times B$) there exists a basis element $(a, b) \times (c, d)$ such that $0 \times 0 \in (a, b) \times (c, d) \subset W$. To prove that W intersects $F^{-1}C$, prove that $(a, b) \times (c, d)$ intersects $\{x \times x | x \in \mathbb{R} \text{ and } x \neq 0\}$. Sketch a picture of these two sets to help yourself figure out how to write that down. You might want to write a conclusion to this argument, summarizing what you did, since it had a complicated set up.