

Topology I, Newberger, Spring 2005

Homework: Sections 14,15,16. Due Tuesday, February 15th.

Remark:

When you want to prove a set C is not open, you find a point $x \in C$, and show that there is not a basis element B such that $x \in B \subset C$. It is not, in general clear to simply make the claim that no such basis element exists. However, the statement often becomes clear if you say instead that any basis element B that contains x fails to lie in C . This can be made clearer yet by writing B in the form that is dictated by the particular topology in question.

Tips:

I.B. Let X and Y be topological spaces, and consider $X \times Y$ as a topological space with the product topology. Assume that U is an open set in $X \times Y$. What do you know about U ?

Two useful statements about U can be made. The first is that for every point $x \times y$ in U , there exists a basis element B for the product topology on $X \times Y$ such that $x \times y \in B \subset U$. Since the product topology is generated by the basis $\{V \times W \mid V \text{ is open in } X \text{ and } W \text{ is open in } Y\}$, we have the following statement: for every $x \times y \in U$, there exist open sets $V \subset Y$ and $W \subset X$ such that $x \times y \in V \times W \subset U$. Note that U is not necessarily equal to a product of open sets.

The second useful statement you can make is that U is a union of basis elements for the product topology on $X \times Y$. This means that

$$U = \bigcup_{\lambda \in \Lambda} V_\lambda \times W_\lambda$$

where Λ is some index set, and for each $\lambda \in \Lambda$, the set V_λ is open in X and the set W_λ is open in Y .

I.C. Do problem #4 on page 92.

You want to prove that if U is open in $X \times Y$, then $\pi_1(U)$ is open in X . You need not repeat the argument for π_2 . Begin by letting U be an open set in $X \times Y$ and use the second useful statement listed above to write $U = \bigcup_{\lambda \in \Lambda} V_\lambda \times W_\lambda$ where Λ is some index set, and for each $\lambda \in \Lambda$, the set V_λ is open in X and the set W_λ is open in Y . Apply π_1 to U . Use problem #3f on page 21 and the fact that $\pi_1(A \times B) = A$ to finish the argument.

IV. Problem #9 page 92.

First consider $\mathbb{R} \times \mathbb{R}$ with the dictionary topology. Denote the topology on this set by \mathcal{T}_{dict} , and the basis for this order topology by \mathcal{B}_{dict} . Now consider $\mathbb{R} \times \mathbb{R}$ with the product topology arising from the discrete topology in the first coordinate and the standard topology in the second coordinate. Call this topology $\mathcal{T}_{d \times std}$, and call the basis $\mathcal{B}_{d \times std}$. Explain your notation on your homework.

We want to prove that these two topologies are equal. Lemma 13.3 says that it is enough to prove that the basis for one topology is contained in the other topology

and vice versa. In fact, we will prove that $\mathcal{B}_{dict} \subset \mathcal{T}_{d \times std}$ and $\mathcal{B}_{d \times std} \subset \mathcal{B}_{dict}$ (which is a subset of \mathcal{T}_{dict}).

To prove $\mathcal{B}_{d \times std} \subset \mathcal{B}_{dict}$, begin by letting $B \in \mathcal{B}_{d \times std}$. Then $B = \{x\} \times (a, b)$ where $x, a, b \in \mathbb{R}$ and $a < b$. This is because the basis for the discrete topology on \mathbb{R} is $\{\{x\} | x \in \mathbb{R}\}$, and the basis for the standard topology on \mathbb{R} is $\{(a, b) | a < b, a, b \in \mathbb{R}\}$. Sketch a picture of the set B in the margin of your homework. Rewrite B as an interval in the dictionary topology. Since the basis elements for \mathcal{B}_{dict} are intervals, this set will belong to \mathcal{B}_{dict} .

To prove that $\mathcal{B}_{dict} \subset \mathcal{T}_{d \times std}$, let $B \in \mathcal{B}_{dict}$. Then $B = (a \times b, c \times d)$. Sketch a picture of this set in the margin of your homework. There are two possible shapes that this set can take. One is a vertical finite open ended line segment, while the other is a vertical strip with part of its boundary. Prove that B is in $\mathcal{T}_{d \times std}$ by proving that for every $x \times y \in B$, there exists $C \in \mathcal{B}_{d \times std}$ that contains $x \times y$ such that $C \subset B$.

So start by saying let $x \times y \in B$. The $C \in \mathcal{B}_{d \times std}$ that you want to find will be a vertical finite open ended interval containing $x \times y$ that lies entirely in B . Say what C is explicitly in terms of B and x and y . Draw pictures of possible B 's, $x \times y$'s and C 's to clarify your argument.

IV. Problem #10 on page 92.

I made a mistake when grading this problem, and decided to drop it. If you worked on it and got some parts correct, I gave extra credit for it. Ask me if you want to know the answer.