Topology I, Newberger, Spring 2005

Homework: Sections 7,12 and 13 Tips and Remarks.

III. B. Here are some methods for showing a set is countable and some for uncountable. Do not refer to these tips in your write-up.

Countable Tips

(i) A surjective function mapping a set that you know is countable to B.
(ii) An injective function mapping B to a set you know is countable.
(iii) Decompose B into a countable union of countable sets.
(iv) Show that B is a subset of a set that you have already proved or know is countable.

Uncuntable Tips

(i) Construct a bijection from B to $\{0,1\}^{\omega}$, that was proved to be uncountable in Theorem 7.7.
(ii) Construct an injection from $\{0,1\}^{\omega}$ to B. This is the same as proving that B has an uncountable subset, since the injection is a bijection from the uncountable set to its image. Thus B must be uncountable, since countable sets do not have uncountable subsets, by Corollary 7.3.
(iii) Use the idea found in the proof of Theorem 7.7. To do this, consider an arbitrary function $g : \mathbb{Z}_+ \to B$.Show that g cannot be surjective, by explicitly finding an element of B that is not in the image of g. Read the proof of Theorem 7.7 for more insight.
(iv) By Theorem 7.8, the power set of a countable set is uncountable. So you can prove B is uncountable by constructing a bijection from B to the powerset of a countable set.

Remark

To define a function Φ from a set X to a set Y, you must begin with an arbitrary element $x \in X$ and say what element of Y you mean when you say $\Phi(x)$.

Here are tips for each part of problem 5 on page 51.

(a) Countable.

Idea 1 (nice because it reinforces material in the text) Use Countable Tip (ii). Read the proof of Corollary 7.4. Note that all the information about each f in A is determined by two numbers: $f(0)$ and $f(1)$. Create an injective map from A to $\mathbb{Z}_+$. Note that to define $\Phi : A \to \mathbb{Z}_+$, you must take a function $f \in A$, and say what integer $\Phi(f)$ corresponds to.

Idea 2 (nice because it is simple) Use Countable Tip (ii). Create an injective map from A into $\mathbb{Z}_+ \times \mathbb{Z}_+$. Note that to define $\Phi : A \to \mathbb{Z}_+$, you must take a function $f \in A$, and say what pair of integers in $\mathbb{Z}_+ \times \mathbb{Z}_+$ the answer $\Phi(f)$ corresponds to.

(b) Countable. Generalize your ideas from part (a).

(c) Countable. Use Countable Tip (iii). If you do (b), you get this for free, by Theorem 7.5. If you don't do (b), you have to prove this one explicitly.

(d) Uncountable.

Idea 1 Use Uncountable Tip (ii). Construct a bijection from the set E, given in part (e) to $\{0,1\}^\omega$. Then show that the set E is a subset of the set D.

Idea 2. Use Uncountable Tip (iii).

(e) Uncountable. Use Idea 1 in part (d).

(f) Countable. Use Countable Tip (iii). Consider the sets

$$F_n = \{f \in F | f(k) = 0 \text{ for all } k > n\}.$$

Prove for each n, this set is finite, and that $F = \bigcap_{n\in\mathbb{Z}_+} F_n$.

(g) Countable. See part (f).

(h) Countable. Use Countable Tip (iii). Decompose H into 10 sets similar to those in (f) and (g) above.

(i) Countable. Use Countable Tip (iii). Notice that if $\{n, m\}$ is a two element set, then $m \neq n$. Consider the sets

$$I_n = \{\{n, m\} | m < n\}.$$

(So when $n = 3$, $I_3 = \{\{3, 2\}, \{3, 1\}, \{3, 0\}\}$.) Show that each I_n is finite, and that $I = \bigcup_{n\in\mathbb{Z}_+} I_n$.

(j) Countable. Use Countable Tip (iii). Decompose J into sets of the form

$$J_n = \{\text{all subsets of } \mathbb{Z}_+ \text{ with } n \text{ elements}\}.$$

Then prove each J_n is countable similarly to the proof of part (i).

V.A.
Tips for showing that a topology $\mathcal{T}_1$ is **not contained** in a topology $\mathcal{T}_2$ (i.e for showing $\mathcal{T}_1 \nsubseteq \mathcal{T}_2$) when $\mathcal{T}_1$ is generated by a basis $\mathcal{B}_1$ and $\mathcal{T}_2$ is generated by a basis $\mathcal{B}_2$.

(i) **Idea:** Prove that there exists an element $U \in \mathcal{T}_1$ such that $U \notin \mathcal{T}_2$.

(ii) **More specifically:** Prove that there is a basis element $B_1 \in \mathcal{B}_1$, such that $B_1 \notin \mathcal{T}_2$. This is just a special case of the idea, since $\mathcal{B}_1 \subset \mathcal{T}_1$ (i.e. since basis elements are open).

(iii) **More specifically:** How are you going to show that B_1 is not an element of $\mathcal{T}_2$? Since $\mathcal{T}_2$ is defined using the basis $\mathcal{B}_2$, we show a set $U \in \mathcal{T}_2$ by showing

for every $x \in U$, there exists $B_2 \in \mathcal{B}_2$ containing x such that $B_2 \subseteq U$.

Negate this statement to figure out what you need to do to show the set B_1 is not contained in $\mathcal{T}_2$: Show $B_1 \notin \mathcal{T}_2$ by showing

there exists $x \in B_1$ such that for every $B_2 \in \mathcal{B}_2$ that contains x, we have $B_1 \nsubseteq U$.

(iv) **More specifically:** So begin by choosing a basis element $B_1 \in \mathcal{B}_1$ and a specific point $x \in B_1$ that you think will work. Then prove the for every statement in part (iii): it says for every $B_2 \in \mathcal{B}_2$ that contains x... To prove this for every statement, start with: Let $B_2 \in \mathcal{B}_2$ be a basis element that contains x. Now show that the set $B_2 \nsubseteq B_1$ by finding a point $a \in B_2$ such that $a \notin B_1$.

V.C. Problem 6 on page 83. If it will help you organize the information, summarize the explanation about how to proceed given in V.A. before beginning your proof. Certainly there is no need to copy it word for word, but sometimes it is helpful to explain ideas back to yourself before applying them.

(i) Prove $\mathbb{R}_K \not\subseteq \mathbb{R}_l$. Start by choosing a basis element of $B_K \in \mathcal{B}_K$ and an $x \in B_K$. Here the issues arise from sets containing 0. I suggest starting with a set $B_K = (-1, 1) - K \in \mathcal{B}_K$, and letting $x = 0$. Now let $B_l \in \mathcal{B}_l$ be a basis element that contains 0. Then $B_l = [a, b)$ for some $a < b$. Use the fact that $0 \in [a, b)$ to show that there is a number $y \in [a, b)$ that is not an element of $(-1, 1) - K$. Conclude that for all basis elements $B_l \in \mathcal{B}_l$ that contain 0, $B_l \not\subseteq (-1, 1) - K$.

(ii) Prove $\mathbb{R}_l \not\subseteq \mathbb{R}_K$. Start by choosing a basis element of $B_l \in \mathcal{B}_l$ and an $x \in B_l$. Here the issues arise from the left end point of the basis elements. I suggest starting with a set $B_l = [2, 3) \in \mathcal{B}_l$, and letting $x = 2$. Any choice of $B_l \in \mathcal{B}_l$ will work, as long as x is the left end point. Now let $B_K \in \mathcal{B}_K$ be a basis element that contains 2. Then $B_K = (a, b)$ or $B_K = (a, b) - K$ for some $a < b$. This can be made more efficient, but I suggest treating these as two different cases, and proving the statement in each cases.

Case 1. Suppose $B_K = (a, b)$. Use the fact that $2 \in (a, b)$ to show that there is a number $y \in (a, b)$ that is not an element of $[2, 3)$.

Case 2. Suppose $B_K = (a, b) - K$. Use the fact that $2 \in (a, b) - K$ to show that there is a number $y \in (a, b) - K$ that is not an element of $[2, 3)$.

Conclude that for all basis elements $B_l \in \mathcal{B}_l$ that contains 2, $B_l \not\subseteq [2, 3)$.

VI. Problem 7 on page 83. In this part we have

$$\mathcal{T}_3 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subsetneq \mathcal{T}_4$$

and

$$\mathcal{T}_5 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subsetneq \mathcal{T}_4$$

The topologies $\mathcal{T}_3$ and $\mathcal{T}_5$ are not comparable.

You should redo the parts that have an $\times$ next to them on your paper. If there is a check, you need not rewrite that part. Here I will concentrate on containment. Use the methods described in V to prove that the containment is proper.

Use $\mathcal{B}_i$ to denote the basis of $\mathcal{T}_i$, for $i = 1, 2, 3, 4, 5$.

(a) $\mathcal{T}_5 \subseteq \mathcal{T}_1$. Show that elements of the basis for $\mathcal{T}_5$ are unions of basis elements from $\mathcal{B}_1$, and hence open in $\mathcal{T}_1$.

(b) $\mathcal{T}_3 \subseteq \mathcal{T}_1$. Show that elements of the basis for $\mathcal{T}_3$ are unions of basis elements from $\mathcal{B}_1$, and hence open in $\mathcal{T}_1$.

(c) $\mathcal{T}_1 \subseteq \mathcal{T}_2$ by Lemma 13.4.

(d) $\mathcal{T}_2 \subseteq \mathcal{T}_4$. This one is deceiving. The topology $\mathcal{T}_4$ is different from the topology on $\mathbb{R}_l$, when comparing with $\mathbb{R}_K$ (i.e. the topology $\mathcal{T}_2$). First show that basis elements for $\mathcal{T}_2$ of the form (a, b) are open in $\mathcal{T}_4$ (by showing that (a, b) is a union of basis elements from $\mathcal{T}_4$, and so are open in $\mathcal{T}_4$, or by using a method similar to that in Lemma 13.4). Then show that basis elements for $\mathcal{T}_2$ of the form $(a, b) - K$ that contain 0 are a union of open sets in $\mathcal{T}_2$. Hint: $(a, 0]$ will be one of them.