**Topology I, Newberger, Spring 2005**

**Homework: Sections 1-6 Tips and Remarks.** IB1.

i. **About the notation $f^{-1}$.** These problems involve a function $f : A \to B$, and inverse images under this function. Since $f^{-1}$ is not a function in general, be careful not to use function notation with the symbol $f^{-1}$. For example, it is not appropriate to write $f^{-1} \circ f$ since $\circ$ is reserved for composition of functions. Even though $f^{-1}$ is not a function, we do sometimes use the notation $f^{-1}(y)$, where $y \in B$, however, but it denotes a subset of $A$ rather than a single element of $A$. For a set $S \subset B$, we have

$$f^{-1}(S) = \{x \in A | f(x) \in S\}.$$

In particular, for an element $y \in B$, we have

$$f^{-1}(y) = \{x \in A | f(x) = y\}.$$

So in the course of your proof, if you have $x \in f^{-1}(S)$, then you can conclude from the definition that $f(x) \in S$. Going from $x \in f^{-1}(S)$ to $f(x) \in S$ is due directly to the definition of $f^{-1}(S)$, rather than to an application of $f$ to both sides of $x \in f^{-1}(S)$. Similarly, if you have $f(x) \in S$, you can conclude from the definition that $x \in f^{-1}(S)$. This does not follow from an application of "$f^{-1}$" (which is not a function, so cannot be *applied* per se) to both sides of of $f(x) \in S$.

Be careful not to treat $f^{-1}$ as a function, unless $f$ is a bijection, and not to treat $f^{-1}(y)$ as a single element, unless $f$ is one-to-one.

ii. **Setting up 2(c).** To prove that $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$, You must prove that

$$f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$$

and

$$f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1).$$

To prove that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$, you must show that for every $x \in f^{-1}(B_0 \cap B_1)$, we have $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. This is a statement that you want to prove "for every" element of $f^{-1}(B_0 \cap B_1)$. To prove this for every statement, you should begin by choosing an arbitrary element of $f^{-1}(B_0 \cap B_1)$. Then prove the statement holds for that element, and at the end of your argument, explain that because the element was chosen arbitrarily, the proof works for all of the elements of $f^{-1}(B_0 \cap B_1)$.

Thus begin with the statement "Let $x \in f^{-1}(B_0 \cap B_1)$." Now use the definition of $f^{-1}$ described above to conclude that $f(x) \in B_0 \cap B_1$. Continue, using the definition of $\cap$ and $f^{-1}$ to explain why $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$.

Likewise, to prove that $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$, begin with "Let $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$." Then you will have that $x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$. Use the definitions of $f^{-1}$, described above, and of intersection to show that $x \in f^{-1}(B_0 \cap B_1)$.

iii. **Setting up 2(g)** For this problem you need to prove two separate things:
(1) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.
(2) If $f$ is injective, then $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$.

Note that when $f$ is injective, you have both inclusions, so proving these two things implies that when $f$ is injective, $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$.

(1) First, prove that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$. As above, you need to prove that *for every* $x \in f(A_0 \cap A_1)$, we have $x \in f(A_0) \cap f(A_1)$. So begin with an arbitrary element of $f(A_0 \cap A_1)$, i.e. start by saying "Let $y \in f(A_0 \cap A_1)$." (I used $y$ because the set $f(A_0 \cap A_1)$ is a subset of the codomain $B$ of $f$, and the variable $y$ seems more intuitive than the variable $x$.)

Now since $f^{-1}$ is not a function, it does not make sense to "take $f^{-1}$ of both sides." However, you can use the definition of $f(A_0 \cap A_1)$. We have

$$f(A_0 \cap A_1) = \{f(x) | x \in A_0 \cap A_1\}.$$

So $y \in f(A_0 \cap A_1)$ implies that there exists $x \in A_0 \cap A_1$ such that $f(x) = y$. Now you have an $x \in A_0 \cap A_1$ to work with. We have

$$f(A_0) = \{f(x) | x \in A_0\} \quad \text{and} \quad f(A_1) = \{f(x) | x \in A_1\}.$$

So to show that $y \in f(A_0) \cap f(A_1)$, you must show that $y \in f(A_0)$ and $y \in f(A_1)$. By definition of these sets, this means you must show that there is an element $x_0 \in A_0$ such that $f(x_0) = y$, and an element $x_1 \in A_1$ such that $f(x_1) = y$.

(2) Now suppose that $f$ is injective. You must prove that $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$. As above, you need to prove that *for every* $x \in f(A_0) \cap f(A_1)$, we have $x \in f(A_0 \cap A_1)$. So start with an arbitrary element of $f(A_0) \cap f(A_1)$. This means start by saying, "Let $y \in f(A_0) \cap f(A_1)$." Now that gives you $y \in f(A_0)$ and $y \in f(A_1)$. We have

$$f(A_0) = \{f(x) | x \in A_0\} \quad \text{and} \quad f(A_1) = \{f(x) | x \in A_1\}.$$

By definition of these sets, $y \in f(A_0)$ means that there is an element $x_0 \in A_0$ such that $f(x_0) = y$, and $y \in f(A_1)$ means that there is an element $x_1 \in A_1$ such that $f(x_1) = y$. Use the fact that $f$ is injective to show that $x_0 = x_1$. (Appeal directly to the formal definition of injective.) Finish the problem using the definition of the set $f(A_0 \cap A_1)$.

**Upshot:**

$$y \in f(A) \Leftrightarrow \exists x \in A \text{ such that } f(x) = y,$$

and

$$x \in f^{-1}(B) \Leftrightarrow f(x) \in B.$$