Newberger: Math 233, Spring 2004

Worksheet: Functions.

(1) (10 points) For any natural number n, let $\mathbb{N}_n = \{1, 2, ..., n\}$.
   (a) List all functions $f : \mathbb{N}_3 \to \mathbb{N}_3$. You do not need to find a formula for each function; instead just give the image of each element of $\mathbb{N}_3$ is under your function. One example of a function $f : \mathbb{N}_3 \to \mathbb{N}_3$ is given by $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$. Be systematic.
   (b) Which of the functions you listed are constant? Which one is the identity function?
   (c) How many functions are there from $\mathbb{N}_n$ to $\mathbb{N}_n$? Explain how you came up with that number.
   (d) How many constant functions are there from $\mathbb{N}_n$ to $\mathbb{N}_n$?

Definition: For the next two exercises, you will need the following definition: We say two functions $f, g : A \to B$ *commute* if $f \circ g = g \circ f$.

(2) (10 points) Let $a_1, a_2 \in \mathbb{R}$ and let $f_1 : \mathbb{R} \to \mathbb{R}$ be the constant function $f_1(x) = a_1$, and let $f_2 : \mathbb{R} \to \mathbb{R}$ be the constant function $f_2(x) = a_2$. Prove that f_1 and f_2 commute if and only if $a_1 = a_2$. Recall that to prove an if and only if statement, you must do two proofs:
   (a) If f_1 and f_2 commute, then $a_1 = a_2$.
   (b) If $a_1 = a_2$, then f_1 and f_2 commute.

(3) (10 points)
   (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 4x + 6$ and $g(x) = 3x + 4$. Show that f and g commute.
   (b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax + b$ and $g(x) = cx + d$. Find a condition on $a, b, c, d \in \mathbb{R}$ that will guarantee that f and g commute. You should be able to prove that if your condition holds, then f and g commute. Use sentences.
   (c) Does your condition hold for the functions in part (a)? In other words, is the statement, "if your condition holds, then f and g commute," a generalization of the statement "if $f(x) = 4x + 6$ and $g(x) = 3x + 4$, then f and g commute"? Explain.