Newberger: Math 233, Spring 2004

# Worksheet: Closed under + and $*$.

(1) (10 points) Prove the rational numbers $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z} \text{ and } q \neq 0\}$ are closed under both addition and multiplication.

(2) (10 points) Prove that the set $W = \{\frac{m}{3^n} | m \in \mathbb{Z}, n \in \mathbb{N}\}$ is closed under both addition and multiplication.

(3) (10 points) Consider the set $2\mathbb{Z} = \{2a | a \in \mathbb{Z}\}$.
- Prove $2\mathbb{Z}$ is closed under addition.
- Was the number 2 important in this? Conjecture a generalization of this statement.
- Prove your generalization.

(4) (10 points) Consider the set $L = \{a + ai | a \in \mathbb{R} \text{ and } i^2 = -1\}$.
- Prove that $L$ is closed under addition.
- Prove that $L$ is not closed under multiplication. To do this give a numerical counterexample to the statement "$L$ is closed under multiplication." In other words, you should give a counterexample to the statement:

  *If $x, y \in L$, then $xy \in L$.*

  Remember that a counterexample to a statement is something that satisfies the assumptions but not the conclusion. Here you should name two specific complex numbers $x$ and $y$ that belong to $L$ (so they satisfy the "if"). Then you should calculate the product $xy$, and explain how you can tell that $xy$ is not in $L$ (so the "then" is not satisfied).

  Tip: In this exercise, I ask for a numerical counterexample, not a general explanation. In Exercise (5), you will have the opportunity to give more insight into the multiplication of complex numbers.

(5) (10 points) Again consider the set $L = \{a + ai | a \in \mathbb{R} \text{ and } i^2 = -1\}$. In this exercise, you will make a conjecture about what happens when two points in $L$ are multiplied.

The set of complex numbers is given by $\mathbb{C} = \{a + bi | a, b \in \mathbb{R} \text{ and } i^2 = -1\}$. Let $z \in \mathbb{C}$. Then there are numbers $a, b \in \mathbb{R}$ such that $z = a + ib$. We can visualize the complex number $z = a + ib$ in $\mathbb{R}^2$ by plotting the point $(a, b)$ (i.e. with $x$-coordinate $a$ and $y$-coordinate $b$). For example The complex number $2 + 5i$ would be plotted with $x = 2$ and $y = 5$ in $\mathbb{R}^2$.
- Draw the points in $\mathbb{R}^2$ that correspond to the set $L$.
- On your graph, clearly label the points $x, y \in L$ that you used to show that $L$ is not closed under multiplication. Also label the point $xy$.
- To get intuition, calculate the product of at least three more pairs of points from $L$. Plot these on your graph (make a new graph if your original one would get too cluttered).
- Make a conjecture about the product of two elements in $L$. For example, your statement could begin "If $x, y \in L$, then $xy$..."
- Prove your conjecture.

(6) (10 points) Let $z \in \mathbb{C}$. Then there are real numbers $a$ and $b$ such that $z = a + bi$. The modulus of $z$, denoted $|z|$, is the non-negative real number $|z| = \sqrt{a^2 + b^2}$.
- Calculate the modulus of the following complex numbers: $2 + 5i$, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, $a + ai$ (where $a \in \mathbb{R}$). Label your answers ("$|2 + 5i| = ...$").
- Prove that the set $S^1 = \{a + ib \in \mathbb{C} \big| |a + bi| = 1\}$ is closed under multiplication.