

**Section 2.10 #4** Prove that for every integer  $n \geq 0$ , the number  $n^4 - 4n^2$  is divisible by 3.

Begin by stating what  $P(n)$  is on your homework. In this case, begin with:

*Let  $P(n)$  be the statement “ $n^4 - 4n^2$  is divisible by 3.”*

Basis step: Since in this problem, you want to prove  $P(n)$  is true for all  $n \geq 0$ , the base case is  $P(0)$ . Sometimes it is difficult to know what to write when the statement you are supposed to prove is trivial. In this case, I suggest:

*Basis step:  $P(0)$  is the statement “ $0^4 - 4(0)^2$  is divisible by 3,” which is true since  $3|0$ .*

Induction step: Assume that  $P(k)$  is true, and prove that  $P(k + 1)$  is true. In your proof say that you are assuming  $P(k)$ , and what it means.

*Induction step: We assume that  $P(k)$  holds, which means that  $3|(k^4 - 4k^2)$ . This means we can find an integer  $c$  such that  $k^4 - 4k^2 = 3c$ .*

It may help you to organize your argument to write that you want to prove  $P(k + 1)$ , and what it means.

*We want to prove  $P(k + 1)$ , so we want to show that  $(k + 1)^4 - 4(k + 1)^2$  is divisible by 3. This means we want to find an integer  $d$  such that  $(k + 1)^4 - 4(k + 1)^2 = 3d$ .*

Now you should work with the expression  $(k + 1)^4 - 4(k + 1)^2$  and try to manipulate it into 3 times something. Begin by multiplying it out:

$$(k + 1)^4 - 4(k + 1)^2 = k^4 + 4k^3 + 2k^2 - 4k - 3$$

We want to use the induction hypothesis  $k^4 - 4k^2 = 3c$ , so we are looking for a term with  $k^4 - 4k^2$ . To get this, subtract and add  $4k^2$ .

$$\begin{aligned} k^4 + 4k^3 + 2k^2 - 4k - 3 &= k^4 + (-4k^2 + 4k^2) + 4k^3 + 2k^2 - 4k - 3 \\ &= (k^4 - 4k^2) + 4k^3 + (4k^2 + 2k^2) - 4k - 3 \end{aligned}$$

Now substitute in  $3c$  for  $k^4 - 4k^2$ , and factor out as many 3's as possible.

$$\begin{aligned} (k^4 - 4k^2) + 4k^3 + (4k^2 + 2k^2) - 4k - 3 &= 3c + 4k^3 + 6k^2 - 4k - 3 \\ &= 3(c + 2k^2 - 1) + 4k^3 - 4k \end{aligned}$$

This is a good start, but to finish, you need to prove  $4k^3 - 4k$  is divisible by 3. Do this by induction, as a lemma. Write something like:

*We will use induction to show that  $4n^3 - 4n$  is divisible by 3 for any integer  $n \geq 0$ . Let  $P(n)$  be the statement  $4n^3 - 4n$  is divisible by 3.*

The set up for this induction problem is similar to that of the main exercise, and I am going to leave it to you to work through in your homework. Once you have done the proof that  $4n^3 - 4n$  is divisible by 3 for all integers  $n$ , return to your main argument:

*Since  $4k^3 - 4k$  is divisible by 3, there is an integer  $f$  such that  $4k^3 - 4k = 3f$ . Thus*

$$\begin{aligned}(k+1)^4 - 4(k+1)^2 &= 3(c + 2k^2 - 1) + 4k^3 - 4k \\ &= 3(c + 2k^2 - 1) + 3f \\ &= 3(c + 2k^2 - 1 + f).\end{aligned}$$

*Let  $d = c + 2k^2 - 1 + f$ . Then  $(k+1)^4 - 4(k+1)^2 = 3d$  as desired.*

Write a sentence in conclusion saying that by induction, you have completed the proof.