Newberger: Math 233, Spring 2004

Activity: Limits tending to infinity.

1. Guessing about neighborhoods of infinity.
   a. Let $L \in \mathbb{R}$ and let $\delta > 0$. Draw a diagram representing the set $S = \{x \in \mathbb{R} |\ |x-a| < \delta\}$ on the number line. What is the center of this interval? What is its radius?
   b. The set S given here is called the δ-neighborhood of a. Is 3 in the .3-neighborhood of π? Give an example of a number in the 2-neighborhood of 0.
   c. What would a neighborhood of $+\infty$ look like? Draw a diagram representing how you think a neighborhood of infinity should appear.

2. Let $f(x) = \frac{6}{x} + 2$.
   a. Draw a graph of the function f.
   b. Draw the set $S = \{y \in \mathbb{R} |\ |y-2| < 1\}$ on the y-axis. (This is the 1-neighborhood of 2.)
   c. Name a number x so that $f(x)$ is in the 1-neighborhood of 2.
   d. Draw the arrows for $x = 1$, $x = 2$ and $x = 3$. Recall that "Draw the arrows for x" means draw an arrow from $(x, 0)$ to $(x, f(x))$ and then another arrow from $(x, f(x))$ to $(0, f(x))$.
   e. How big does x need to be in order for $f(x)$ to be in the 1-neighborhood of 2 (i.e. in S)?
   f. Find a number K such that if $x > K$, then $f(x)$ is in the 1-neighborhood of 2.
   g. For the K you found, prove the if-then statement: if $x > K$, then $f(x)$ is in the 1-neighborhood of 2. Be careful.

3. Again consider function $f(x) = \frac{6}{x} + 2$.
   a. Let $\varepsilon > 0$. Using the previous exercise to guide you, find a number K such that if $x > K$, then $f(x)$ is in the ε-neighborhood of 2. In other words, find K so that if $x > K$, then $f(x)$ is in the set $S = \{y \in \mathbb{R} |\ |y-2| < \varepsilon\}$.
   b. Write a proof that for every $\varepsilon > 0$ there exists K such that if $x > K$ then $|f(x) - 2| < \varepsilon$. First figure out what K should be on scratch paper. Now the statement we are proving is a "for every"-statement, so begin the proof by choosing an arbitrary positive number ε; say "Let $\varepsilon > 0$." Now say "Let K=" whatever you figured out. Prove the if-then statement "If $x > K$, then $|f(x) - 2| < \varepsilon$," using your value of K.
   c. Compare the definition of a limit when x approaches a finite number a to when x approaches $+\infty$.

      Definition: We call a number L the limit of f as x approaches $+\infty$ if for every number $\varepsilon > 0$ there exists a number K such that if $x > K$, then $|f(x) - L| < \epsilon$.

      Definition: We call a number L the limit of f as x approaches a if for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$.
   d. Why does the definition of the limit for a finite a fail to make sense when $a = +\infty$?