Newberger: Math 233, Spring 2004

Activity: Negating Limits.

1. Negating the definition of a limit.
Let (a, b) be non-empty interval in $\mathbb{R}$, and let $c \in (a, b)$. Let $f : (a, b) \to \mathbb{R}$. From the definition of a limit, we know that

   $L = \lim_{x\to c} f(x)$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

   a. When we talked about the symbol $\forall$ originally, we always referred to a set. For example, we said $\forall x \in X$, where X was a set. Here we only write $\forall\varepsilon > 0$. Find a set X so that saying "$\forall\varepsilon > 0$" is equivalent to saying "$\forall\varepsilon \in X$."
   b. Let X be the set you defined above. Let P be the statement "for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$." Write the statement

      For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every x satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

      using only the symbols

      $$P, \quad X, \quad \varepsilon, \quad \delta, \quad \exists, \quad \in, \quad \forall$$

      and the words "such that."
   c. Negate the statement you wrote in the previous part (keeping it in terms of P). Reduce it as much as you can so that the symbol $\sim$ appears neither on the "$\forall$" nor on the "$\exists$."
   d. Write it again using the symbols $\varepsilon > 0$ instead of $\varepsilon \in X$, and $\delta > 0$ instead of $\delta \in X$.
   e. Negate P. In other words, find $\sim P$.
   f. Combine your answer from above to say what you have to do to prove that $\lim_{x\to c} f(X) \neq L$.
   g. Check your answers with another group.

2. Application. Let

   $$f(x) = \begin{cases} 1 & \text{if } x = 2 \\ 4 & \text{if } x \neq 2. \end{cases}$$

   a. What is $f(2)$?
   b. Draw a graph of f (big enough).
   c. On a separate diagram, draw the set $\{(x, y) \in \mathbb{R}^2 |\ |y - f(2)| < 5\}$ and the set $\{(x, y) \in \mathbb{R}^2 | y = 4\}$. Do these sets intersect?
   d. Find a number $\varepsilon > 0$ such that $\{(x, y) \in \mathbb{R}^2 |\ |y - f(2)| < \varepsilon\} \cap \{(x, y) \in \mathbb{R}^2 | y = 4\}$ is the empty set.
   e. For the value of ε you found in the previous part, draw the set $\{(x, y) \in \mathbb{R}^2 |\ |y - f(2)| < \varepsilon\}$ on the y-axis of the diagram showing the graph of f.
   f. Prove that $\lim_{x\to 2} f(x) \neq f(2)$, using your answer to problem (1).