Newberger MATH 233 Spring 04

Activity: Limits

1. Absolute values and distances.
   a. Using symbolic and graphical representations.
      i. Is $\frac{1}{2} \in \{x \in \mathbb{R} |\ |x-2| < 1\}$?
      ii. Let $\delta > 0$ and let $J_\delta = \{x \in \mathbb{R} |\ |x-2| < \delta\}$. Let $I = J_1$. Draw two number lines, one showing a graph of the set I, and another showing a graph of the set J_δ (where δ is some positive number).
      iii. Draw a number line showing two distinct points a and b. Write a formula for the distance between the two points.
   b. Using words.
      i. Let $a \in \mathbb{R}$ What does $|a|$ represent geometrically ("geometrically" means using distances).
      ii. Let $\varepsilon > 0$. Write a verbal description of the set $\{a \in \mathbb{R} |\ |a| < \varepsilon\}$, using distances.
      iii. Write the set $\{a \in \mathbb{R} |\ |a| < b\}$ as an interval.
2. Compare your answers with another group that has finished this set of questions.

3. Graphical representations of functions. Let $f(x) = \frac{3}{2}x$.
   a. Graph $f(x)$. Make the graph large enough; it should cover about 3 square inches of paper.
   b. Express the graph of f as a set of ordered pairs. Write
   $$Graph\ f = \{...$$
   c. On your graph, label a point c on the x-axis. Draw an arrow from $(c, 0)$ to $(c, f(c))$ and an arrow from $(c, f(c))$ to $(0, f(c))$.

      For this exercise, call the process in 3c, "Drawing the arrows for $x = c$." These arrows graphically represent what you do when you read the value of the function f at the point c off of the graph.
4. a. Draw another copy of the graph of $f(x) = \frac{3}{2}x$.
   b. Let $S = \{y \in \mathbb{R} |\ |y-3| < 1\}$. Draw the set S on the y-axis.
   c. Draw the arrows for $x = 2$.
   d. Draw the arrows for a point $x \neq 2$ so that $f(x)$ lands in the set S that you drew on the y-axis.
   e. Name a point x so that $f(x)$ is in S, and a point x so that $f(x)$ is not in S. (Label your answers.)
   f. Write a verbal description of the set $\{x \in \mathbb{R} | f(x) \in S\}$. Explain in terms of drawing arrows. Draw this set on the x-axis.
   g. Write the set $\{x \in \mathbb{R} | f(x) \in S\}$ as an interval.
   h. How close to 2 does x need to be in order for $f(x)$ to be within a distance 1 of 3?
5. Compare your answers with another group that has finished this set of questions.

6. Continue to let $f(x) = \frac{3}{2}x$, and let $\varepsilon > 0$ be a positive number. Goal: Figure out how close x has to be to 2 in order to guarantee that $f(x)$ lies within a distance ε of 3.
   a. Graphical Representation.
      i. Draw the graph of f showing the set $S = \{y \in \mathbb{R} |\ |y - 3| < \varepsilon\}$ on the y-axis.
      ii. Draw the arrows for $x = 2$.
      iii. Draw the set $\{x \in \mathbb{R} | f(x) \in S\}$ on the x-axis.
      iv. Draw the arrows for the endpoints of that set.
   b. Symbolic (or algebraic) representation.
      i. Continue to let $S = \{y \in \mathbb{R} |\ |y - 3| < \varepsilon\}$. What does it mean to say that $f(x) \in S$?
      ii. Write S as an interval.
      iii. Write down inequalities that represent the statement $f(x) \in S$, i.e. write inequalities representing the statement "*$f(x)$ is between the endpoints of the interval S.*"
      iv. Find all x so that $f(x) \in S$ by solving the inequality.
      v. Express the set $\{x \in \mathbb{R} | f(x) \in S\}$ as an interval using your answer to the previous part.
      vi. How far are the endpoints of that interval from $x = 2$?
      vii. Is the interval centered at 2?
      viii. How close does x have to be to 2 in order to guarantee that $f(x)$ lies within a distance ε of 3?
      ix. Call that distance δ. You can write "Let $\delta =$ your answer."
   c. Verbal representation. Since $\varepsilon > 0$ was arbitrary, you have shown

      > *For every positive number ε, you can find a positive number δ such that if the distance from x to 2 is less than δ, then the distance from $f(x)$ to 3 is less than ε.*

      Write that statement using only the symbols

      $$\forall, \quad \exists, \quad \Rightarrow, \quad |\cdot| \text{ (absolute values)}, \quad <, \quad >, \quad -, \quad \varepsilon, \quad \delta, \quad f \text{ and } x,$$

      the numbers 0, 2 and 3, and the words "such that."
7. Compare your answers with another group that has finished this set of questions.

8. **Definition:** Let $f : (a, b) \to \mathbb{R}$, where (a, b) is an interval. Let $c \in (a, b)$. A number L is called "the limit of $f(x)$ as x approaches a" if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$. In this case we write

$$\lim_{x \to c} f(x) = L.$$

For the function $f(x) = \frac{3}{2}x$, and the point $c = 2$, write a proof that $\lim_{x \to 2} f(x) = 3$, using the definition of the limit. Tips:

- This is a "for every" statement, so start by choosing an arbitrary $\varepsilon > 0$. In this case you should begin with "Let $\varepsilon > 0$."
- You already found $\delta > 0$ so that if $|x - 2| < \delta$, then $|f(x) - 3| < \varepsilon$. Write "Let $\delta =$ your answer."
- Prove the if-then statement:

  If $|x - 2| < \delta$, *then* $|f(x) - 3| < \varepsilon$.

  by manipulating the inequality in the if-part into the inequality in the then-part.
- Tip: $|a| < b \Leftrightarrow -b < a < b$. This allows you to get rid of the absolute values before doing the computation.

9. Let

$$f(x) = \begin{cases} \frac{3}{2}x & \text{if } x \leq 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$$

   a. Draw the graph of f showing the set $S = \{y \in \mathbb{R} |\ |y - 3| < \varepsilon\}$ on the y-axis.
   b. Draw the arrows for $x = 2$.
   c. Draw the set $\{x \in \mathbb{R} | f(x) \in S\}$ on the x-axis.
   d. Express the set $\{x \in \mathbb{R} | f(x) \in S\}$ as an interval.
   e. Is that interval centered at 2? Explain what caused that.
   f. How close does x have to be to 2 in order to guarantee that $f(x)$ lies within a distance ε of 3?
   g. Call that distance δ.
   h. Explain what δ represents in words.
   i. Write a proof that $\lim_{x \to 2} f(x) = 3$, using the definition of the limit.

10. Compare your answers with another group that has finished this set of questions.