Newberger MATH 233 Spring 04

Exam 2 name:________________________

GRADES:

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2._______

3._______

4._______

5._______

1. a. (3 points) State the definition of intersection.

Let A and B be sets. Then the intersection of A and B is the set

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

b. (4 points) State the definition of subset.

Let A and B be sets. We say A is a subset of B or $A \subseteq B$ if every element of A is also an element of B, i.e. $A \subseteq B \Leftrightarrow (\forall x) x \in A \Rightarrow x \in B$.

Note that the symbol $A \subseteq B$ does not stand for a set, like $A \cap B$, $A \cup B$ and $A \times B$ do. Instead, the symbol $A \subseteq B$ is a statement that can be either true or false.

c. (6 points) Let $a, b \in \mathbb{Z}$. Prove that $ab\mathbb{Z} \subseteq a\mathbb{Z} \cap b\mathbb{Z}$.

Let $x \in ab\mathbb{Z}$. We want to show that $x \in a\mathbb{Z} \cap b\mathbb{Z}$. This means we want to show $x \in a\mathbb{Z}$ and $x \in b\mathbb{Z}$. Since $x \in ab\mathbb{Z}$, we have $x = abk$ for some $k \in \mathbb{Z}$. Thus if we let $m = ak$ and $n = bk$ we get $x = mb$ and $x = na$. This implies that $x \in a\mathbb{Z}$ and $x \in b\mathbb{Z}$, so $x \in a\mathbb{Z} \cap b\mathbb{Z}$, as desired.

d. (5 points) Give an example of integers $a, b \in \mathbb{Z}$ so that $a\mathbb{Z} \cap b\mathbb{Z} \not\subseteq ab\mathbb{Z}$, and prove that your answer is correct.

Let $a = 6$ and $b = 4$. Then $ab = 24$. We want to show that $a\mathbb{Z} \cap b\mathbb{Z} \not\subseteq ab\mathbb{Z}$. This means we want to find x such that $x \in 4\mathbb{Z} \cap 6\mathbb{Z}$ but $x \notin 24\mathbb{Z}$. Let $x = 12$. Then $x \in 4\mathbb{Z}$, since $x = 4 \cdot 3$, and $x \in 6\mathbb{Z}$ since $x = 6 \cdot 2$. But $x \notin 24\mathbb{Z}$, since $12 \neq 24k$ for any $k \in \mathbb{Z}$. Thus $a\mathbb{Z} \cap b\mathbb{Z} \not\subseteq ab\mathbb{Z}$.

e. (5 points) Prove that $2\mathbb{Z} \cap 3\mathbb{Z} \subseteq 6\mathbb{Z}$.

Let $x \in 2\mathbb{Z} \cap 3\mathbb{Z}$. Then $x \in 2\mathbb{Z}$ and $x \in 3\mathbb{Z}$. Thus $x = 3k$ for some $k \in \mathbb{Z}$, and $x = 2l$ for some $l \in \mathbb{Z}$. We want to show that $x \in 6\mathbb{Z}$, so we want to find $c \in \mathbb{Z}$ such that $x = 6c$.

I can think of several ways to finish this argument. Here are two of them:

Method 1: We have $x = 3x - 2x$, so substituting $x = 2l$ in for the first x and $x = 3k$ in for the second x, we get $x = 3x - 2x = 3(2l) - 2(3k) = 6l - 6k = 6(l - k)$. Let $c = l - k$. Then $x = 6c$ as desired.

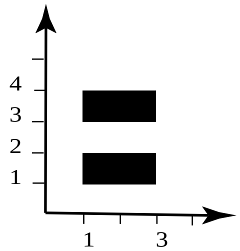
Method 2: We have $x = 2l$ and $x = 3k$. So $2l = 3k$. This implies that $2|3k$. Since 2 is prime, this implies that $2|3$ or $2|k$. Since 2 does not divide 3, we must have $2|k$. So let $k = 2c$ for some $c \in \mathbb{Z}$. Then $x = 3k = 3(2c) = 6c$, as desired.

2. a. (3 points) Define Cartesian product.

Let A and B be sets. The Cartesian product of A and B is the set

$$A \times B = \{(x, y) | x \in A \text{ and } x \in B\}.$$

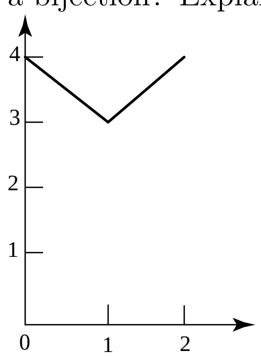
b. (4 points) Is S the Cartesian product of two sets in $\mathbb{R}$? Prove it is not, or name sets A and B in $\mathbb{R}$ such that $A \times B = S$.



The set S

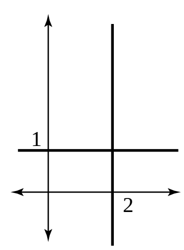
The set S shown here is equal to $A \times B$ where $A = [1, 3]$ and $B = [1, 2] \cup [3, 4]$.

c. (5 points) The diagram below shows the graph of a function $f : [0, 2] \to [2, 4]$. Is f a bijection? Explain your answer.



This function is not a bijection. To explain why, we must either explain that f is not one-to-one or is not onto. The function f is not one-to-one since $f(0) = 3$ and $f(2) = 3$, so $f(0) = f(2)$, while $0 \neq 2$. The function f is not onto $[2, 4]$ since, for example, $f(x)$ is never equal to 2, though 2 is an element of the codomain $[2, 4]$.

d. (4 points) Draw a diagram showing the set $T = (\{2\} \times \mathbb{R}) \cup (\mathbb{R} \times \{1\})$.



The set T includes both the vertical line through 2 and the horizontal line through 1.

3. a. (5 points) State the definition of the limit of a function $f(x)$ as x tends to a finite number.

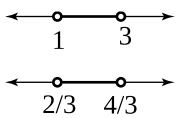
Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let $f : (a, b) \to \mathbb{R}$. Let $c \in (a, b)$. Then we say the limit of $f(x)$ as x approaches c is equal to L if the following holds: for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

b. Let $f(x) = 5 - 3x$.

i. (4 points) Let $S = \{y \in \mathbb{R} | \; |y - 2| < 1\}$. Let $T = \{x \in \mathbb{R} | f(x) \in S\}$. Draw two number lines, one showing S and the other showing T.

To figure out what S is, note that $y \in S$ if and only if $|y - 2| < 1$, which is equivalent to $-1 < y - 2 < 1$, or $1 < y < 3$. So S is the interval $(1, 3)$.

To figure out what T is, note that $x \in T$ if and only if $f(x) \in S$, i.e. if and only if $|f(x) - 2| < 1$, which is equivalent to $-1 < 5 - 3x - 2 < 1$, or $2/3 < x < 4/3$. Thus we have $T = (2/3, 4/3)$.



ii. (6 points) Prove that $\lim_{x \to 1} f(x) = 2$. Clearly label the beginning and end of your proof.

Let $\varepsilon > 0$ and let $\delta = \varepsilon/3$. We will prove the if-then statement:
If $|x - 1| < \delta$, then $|f(x) - 2| < \varepsilon$.
Suppose $|x - 1| < \delta$. Then $-\delta < x - 1 < \delta$, and since $\delta = \varepsilon/3$, we get $-\varepsilon/3 < x - 1 < \varepsilon/3$. This implies that $-\varepsilon < 3x - 3 < \varepsilon/3$. Multiplying through by -1, and remembering to switch the inequalities, we get $-\varepsilon < -3x + 3 < \varepsilon$, which implies $|-3x + 3| < \varepsilon$. But $f(x) - 2 = -3x + 3$, so we have $|f(x) - 2| = |-3x + 3| < \varepsilon$, as desired.

4. a. (5 points) State the definition of one-to-one.

   Let A and B be sets, and let $f : A \to B$. The f is one-to-one if for all x_1 and x_2 in A, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

   b. (5 points) State the definition of onto.

   Let A and B be sets, and let $f : A \to B$. The f is onto if for every $y \in B$, there exists $x \in A$ such that $f(x) = y$.

   c. In each of the following parts, give an example of a function satisfying the given conditions. The domains and codomains of your functions must be chosen from the sets U, V and W given here:

$$U = \{1, 2, 3\}, \quad V = \{4, 5, 6\}, \quad W = \{8, 9\}$$

   i. (3 points) One-to-one, but not onto.

   Let $f : W \to V$ be given by $f(8) = 4$ and $f(9) = 6$. Then f is one-to-one but not onto.

   ii. (3 points) Onto, but not one-to-one.

   Let $f : U \to W$ be given by $f(1) = 8$, $f(2) = 9$ and $f(3) = 8$. Then f is onto but not one-to-one.

   d. (6 points) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$. Prove that f is onto, but $f|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ is not onto.

   First we will prove that f takes $\mathbb{R}$ onto $\mathbb{R}$. Let $y \in \mathbb{R}$. Let $x = y/2$. Then $x \in \mathbb{R}$, and $f(x) = f(y/2) = 2(y/2) = y$. Thus since y was arbitrary, we have for every $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$, namely $x = y/2$, such that $f(x) = y$, and $f : \mathbb{R} \to \mathbb{R}$ is onto.
   Now we will prove that $f|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ is not onto. Note that $3 \in \mathbb{Z}$. Then for all $x \in \mathbb{Z}$, we have $f(x) \neq 3$. One way to see that for all $x \in \mathbb{Z}$, $f(x) \neq 3$ is to note that $f(x)$ is always even, so it cannot be 3, which is odd.

   e. (5 points) Prove that if $f : A \to B$ and $g : B \to C$ are one-to-one, then $g \circ f$ is one-to-one.

   We want to prove $g \circ f$ is one-to-one, so we begin by supposing that $g \circ f(x_1) = g \circ f(x_2)$. We want to prove $x_1 = x_2$. We have $g(f(x_1)) = g(f(x_2))$ (*).
   We know that g is one-to-one, so if we know $g(y_1) = g(y_2)$ for some y_1 and y_2 in B, then we know $y_1 = y_2$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then by (*), we have $g(y_1) = g(y_2)$, so since g is one-to-one, we get $y_1 = y_2$, i.e. we get $f(x_1) = f(x_2)$.
   Now we have $f(x_1) = f(x_2)$. Since f is one-to-one, this implies that $x_1 = x_2$, which is what we wanted.

5. a. (5 points) Define what it means for two functions to commute.

Let A be a set, and let $f : A \to A$ and $g : A \to A$. We say f and g commute if $f \circ g = g \circ f$.
Note that the domain and codomain of f and g all have to be the same in order for the compositions $f \circ g$ and $g \circ f$ to make sense and be equal.

b. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$. Let f, g and h be functions from $\mathbb{R}$ to $\mathbb{R}$ with $g(x) = x^n$, $f(x) = -x$, and $h(x) = ax + b$.

i. (4 points) Give a condition on n that will guarantee that f and g commute. Write your answer as an if-then statement,

If [your condition] holds, then f and g commute.

$f \circ h(x) = f(h(x)) = -x^n$, while $h \circ f(x) = (-x)^n$. If n is odd, then f and h commute. This holds since when n is odd, we have $(-x)^n = -x^n$. Note that a function is called "odd" precisely when it commutes with $f(x) = -x$.

ii. (5 points) Prove that f and h commute if and only if $b = 0$.

This is an if and only if statement. We must either write two proofs, one proving "$\Rightarrow$" and the other proving "$\Leftarrow$", or write a proof in which every step is $\Leftrightarrow$. I will do the latter. f and h commute if and only if $f \circ h(x) = h \circ f(x)$, which is true if and only if $-(ax + b) = a(-x) + b$. Adding ax to both sides we see that this equality holds if and only if $b = -b$, which is equivalent to (adding b to both sides) $2b = 0$, i.e. $b = 0$.

c. (5 points) Let A be a set and let f, g and h be functions from A to A. Prove that if f commutes with g and with h, then f commutes with $g \circ h$.

Suppose that $f \circ g = g \circ f$ and $f \circ h = h \circ f$. We want to show that $f \circ (g \circ h) = (g \circ h) \circ f$. We have

$$RHS = f \circ (g \circ h)$$
$$= (f \circ g) \circ h,$$

since composition of functions is associative,

$$= (g \circ f) \circ h,$$

since $f \circ g = g \circ f$

$$= g \circ (f \circ h),$$

again since composition of functions is associative

$$= g \circ (h \circ f),$$

since $f \circ h = h \circ f$

$$= (g \circ h) \circ f = LHS,$$

using associativity one last time.