Newberger MATH 233 Spring 04

Exam 1

1. The problems below refer to the following statements. Let P be the statement "x is odd," and let Q be the statement "6 does not divide x."

   a. (1 point) Write the statement $P \Rightarrow Q$ in words.

   If x is odd, then 6 does not divide x.

   b. Write the contrapositive of the statement $P \Rightarrow Q$ in words.

   If 6 divides x, then x is not odd.

   c. Prove the statement $P \Rightarrow Q$, using the contrapositive.

   We know 6 divides x. This means $x = 6k$, for some integer k. We want to show that x is not odd, which means we want to show that x is even. We have $x = 6k = 2(3k)$. Thus x is even.

   d. Conjecture a generalization of $P \Rightarrow Q$.

   One correct solution is: If x is odd, then no even number divides x.

   e. State de Morgan's laws.

   Let P and Q be sentential variables. Then De Morgan's Laws state that $\sim (P \wedge Q)$ is equivalent to $\sim P \vee \sim Q$, and $\sim (P \vee Q)$ is equivalent to $\sim P \wedge \sim Q$.

   f. Referring again to the statements P and Q above, use de Morgan's laws to write the statement $\sim (P \vee \sim Q)$ in words.

   $\sim (P \vee \sim Q)$ is equivalent to $\sim P \wedge Q$, which says: x is not odd, and 6 does not divide x.

   g. Give an example of an integer x such that $\sim (P \vee \sim Q)$ is true.

   $x = 4$ is not odd, and 6 does not divide 4.

2. a. State the definition of divides.

Let a and b be integers, with $a \neq 0$. We say $a|b$, or a divides b, if there exists an integer k such that $b = ak$.

b. For each of the following statements answer True or false. If the statement is false provide a counter example. If the statement is true, provide a proof. You may apply theorems proved in class.

i. Let $a \in \mathbb{Z}$. If $3|a^2$, then $3|a$.

True. This uses Theorem 6.26. Since 3 is prime, if $3|pq$, then $3|p$ or $3|q$. We apply this theorem with $p = a$ and $q = a$. Thus we get $3|a^2$ implies $3|a$ (or $3|a$).

ii. Let $n, a \in \mathbb{Z}$. If $n|a^2$, then $n|a$.

False. This statement looks like Theorem 6.26, but the number n here is not known to be prime. Thus the theorem does not apply. In fact this is a false statement. Take $n = 4$ and $a = 6$. Then n does not divide a since 4 does not divide 6. However 4 does divide 36, so $n|a^2$.

iii. Let $c \in \mathbb{Z}$. If $2|(3c)$, then $2|c$.

True. This uses Theorem 6.26. Since 2 is prime, if $2|pq$, then $2|p$ or $2|q$. We apply this theorem with $p = 3$ and $q = c$. Thus we get $2|3c$ implies $2|3$ or $2|c$. Since 2 does not divide 3, we must have $2|c$.

iv. Let $n, c \in \mathbb{Z}$. If $n|(3c)$, then $n|c$.

False. This statement looks like Theorem 6.26, but the number n is not known to be prime. Thus the theorem does not apply. In fact this statement is false. Take n to be 6, and c to be 2. Then $n|3c$ since $6|6$. However n does not divide c since 6 does not divide 2

3. a. State a version of the principle of mathematical induction.

Let N be an integer. Let $P(n)$ be a statement for each $n \geq N$. Suppose that
(i) $P(N)$ is true.
(ii) For all $k \geq N$, we have $P(k)$ implies $P(k+1)$.
Then $P(n)$ holds for all $n \geq N$.

b. Prove that

$$1 + 3 + ... + (2n - 1) = n^2$$

holds for all integers $n \geq 1$.

Let $P(n)$ be the statement that $1 + 3 + ... + (2n-1) = n^2$.
Then P(1) says $1 = 1^2$, which is true.
Let $k \geq 1$. Suppose

$$1 + 3 + ... + (2k - 1) = k^2.$$

We want to show that $P(k+1)$ holds, i.e. we want to show

$$1 + 3 + ... + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2.$$

The LHS of $P(k+1)$ is

$$\begin{aligned} 1 + 3 + ... + (2k - 1) + (2(k + 1) - 1) &= 1 + 3 + ... + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) = (k + 1)^2, \end{aligned}$$

which is the RHS of $P(k+1)$. Thus $P(k+1)$ holds, and by induction, $P(n)$ holds for all $n \geq 1$.

4. a. State the definition of greatest common divisor.

Let a, b be nonzero integers. Then d is the greatest common divisor of a and b if d satisfies the following two statements.
(i) $d|a$ and $d|b$.
(ii) If $c|a$ and $c|b$, then $c|d$.

b. Let $n \in \mathbb{Z}$ with $n > 1$. Prove that the only common divisors of n and $n+1$ are 1 and -1. In other words, let $b \in \mathbb{Z}$ and prove that if $b|n$ and $b|(n+1)$, then $b = 1$ or $b = -1$.

We know $b|n$ and $b|(n+1)$. Then we have $n = kb$ and $n+1 = lb$ for some $k, l \in \mathbb{Z}$. Plug $n = kb$ into $n+1 = lb$. We get $kb+1 = lb$. But this implies that $lb - kb = 1$, or $b(l-k) = 1$. This implies that $b|1$, but since the only divisors of 1 are 1 and -1, we get $b = 1$ or $b = -1$.

c. Prove that if $(a, 12) = 6$ and $4|ac$, then $2|c$.

We know $(a, 12) = 6$. Then $ax + 12y = 6$ for some integers x and y. We also know that $4|ac$. This means $ac = 4m$ for some integer m. Multiply $ax + 12y = 6$ by c. We get $acx + 12cy = 6c$. Substituting $ac = 4m$, we get $4mx + 12xy = 6c$, or $2mx + 6xy = 3c$. So $2(mx + 3xy) = 3c$, so $2|(3c)$. Apply Theorem 6.26. Since 2 is prime, we have $2|3$ or $2|c$. Since 2 does not divide 3, we must have $2|c$.

5. a. A set S is closed under multiplication if the following holds:

For every $x, y \in S$, we have $xy \in S$.

What does it mean for S **not** to be closed under multiplication? Do not use the expression "it is not true that" nor the expression "not every" in your answer.

There exists $x, y \in S$ such that $xy \notin S$.

b. Let $W = \{k\sqrt{3} | k \in \mathbb{Z}\}$.

i. Is $0 \in W$? Explain your answer.

Yes, $0 \in W$, since $0 = k\sqrt{3}$, when $k = 0$, which is an integer.

ii. Is $4 \in W$? Explain your answer.

4 is not an element of W since if $4 = k\sqrt{3}$, then $k = \frac{4}{\sqrt{3}}$, which is not an integer.

iii. List three different members of W.

$-2\sqrt{3}, \sqrt{3}, 23\sqrt{3} \in W$

iv. Prove that W is closed under addition.

Let $a, b \in W$. Then $a = k\sqrt{3}$, and $b = l\sqrt{3}$. So $a+b = (k+l)\sqrt{3}$. Let $m = k+l$. Then m is an integer, and $a + b = m\sqrt{3}$, so $a + b \in W$.

v. Prove that W is not closed under multiplication.

Note that $2\sqrt{3}$ and $\sqrt{3}$ are elements of W. However $2\sqrt{3}\sqrt{3} = 6$, which is not an element of W. So W is not closed under multiplication.

c. Answer true or false.

i. $\{1, 1, 2, 3\} \neq \{1, 2, 3\}$.

False.

ii. $\{1, 2\} = \{2, 1\}$.

True.

iii. $\{3z | z \in \mathbb{Z}\} = \{3x | x \in \mathbb{Z}\}$.

True.