Newberger Math 461 Spring 02

Sample Final Exam

name:______________________________

Remark: This exam does not have any questions asking the students to show they know how to show a transformation is not linear, or a subset is not a subspace. You are likely to have one or both of these types of problems. Please look at the quizzes for practice problems of this type.

1. Let $A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$.

a. (10 points) Reduce the augmented matrix for the system $A\mathbf{x} = \mathbf{b}$ to Reduced Echelon form.

$$\left[\begin{array}{rrr|r} 1 & 2 & 7 & 3 \\ -2 & 5 & 4 & 3 \\ -5 & 6 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 7 & 3 \\ 0 & 9 & 18 & 9 \\ 0 & 16 & 32 & 16 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 7 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

b. (10 points) Write the solution set for the system $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

The solution set satisfies:

$$\begin{aligned} x_1 + 3x_3 &= 1 \\ x_2 + 2x_3 &= 1 \\ x_3 &\text{ is free.} \end{aligned}$$

So the parametric vector form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 + 1 \\ -2x_2 + 1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

c. (7 points) Give a geometric description of the solution set of the system $A\mathbf{x} = \mathbf{b}$

This is a line through $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *and parallel to* $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Tip: Always give as much information as you can. Saying "a line" is not a complete answer in this case, since you can say precisely which line.

2. Let $A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$.

*** *These are the same as those in problem number 1.* ***

a. (6 points) Are the columns of A linearly independent? Why or why not? *From our work on problem 1, we have that*

$$[A|\mathbf{b}] \sim \left[\begin{array}{rrr|r} 1 & 2 & 7 & 3 \\ -2 & 5 & 4 & 3 \\ -5 & 6 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

since A is the coefficient matrix, we can see that A does not have a pivot position in the third column, which means that the homogeneous equation

$A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, and the columns of A are not linearly independent.

b. (6 points) Do the columns of A span $\mathbb{R}^3$? Why or why not?

Referring to the reduced echelon form of A above, we see that A has a row without a pivot, so the columns of A do not span $\mathbb{R}^3$.

c. (6 points) Give a basis for Col A.

Since the first two columns of A are pivot columns, they form a basis for Col A:

$$\text{basis for Col } A\text{: } \left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\}.$$

Tip: Make sure to refer back to the original A matrix to get the basis columns; the pivot columns from the reduced echelon form of A *do not* form a basis for Col A.

d. (6 points) Give a basis for Nul A.

Using our work from above, we see that

$$[A|\mathbf{0}] \sim \left[\begin{array}{rrr|r} 1 & 2 & 7 & 0 \\ -2 & 5 & 4 & 0 \\ -5 & 6 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

so the solution set for the homogeneous equation $A\mathbf{x} = \mathbf{b}$ is

$$\begin{aligned} x_1 + 3x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_3 &\textit{ is free.} \end{aligned}$$

So the parametric vector form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -2x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} x_3.$$

Thus

$$\textit{basis for Nul } A\text{: } \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that first preforms a reflection across the line $x_2 = -x_1$ and then preforms a rotation by 90 degrees in the counterclockwise direction.

a. (5 points) Find the standard matrix for T.

The columns of the standard matrix for T are $T(\mathbf{e}_1)$, and $T(\mathbf{e}_2)$. If we take $\mathbf{e}_1$ and apply the transformation, we get $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. If we take $\mathbf{e}_2$ and apply the transformation, we get $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So the standard matrix is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

b. (5 points) Is T one-to-one? Why or why not?

Since the standard matrix for T has a pivot position in every **column**, *this transformation is one-to-one.*

Tip: It would be wrong to say that the reason that T is one-to-one is that there is a pivot in every row. It is true that there is a pivot in every row, but this is not the reason that the transformation is one-to-one.

c. (5 points) Is T onto? Why or why not?

Since the standard matrix for T has a pivot position in every **row**, *this transformation is onto.*

Tip: It would be wrong to say that the reason that T is onto is that there is a pivot in every column. It is true that there is a pivot in every column, but this is not the reason that the transformation is onto.

d. (5 points) Can a linear transformation that maps $\mathbb{R}^2$ to $\mathbb{R}^3$ be onto? Why or why not?

A linear transformation that maps $\mathbb{R}^2$ to $\mathbb{R}^3$ cannot be onto, since the standard matrix for such a transformation would have 2 columns and 3 rows. This means that the standard matrix has at most 2 pivot positions, but since there are 3 rows, there will never be a pivot position in every row, so the transformation cannot be onto.

e. (5 points) If a linear transformation mapping $\mathbb{R}^n$ to $\mathbb{R}^m$ is onto, then what must be true about m and n?

The standard matrix for an onto linear transformation will have a pivot in every row, so there must be at least as many columns as rows; thus $n \geq m$.

4. Consider the following matrix.

$$A = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

a. (5 points) What is the characteristic polynomial of A?

The characteristic polynomial of A is $\det(A - \lambda I)$. *In this case*

$$A - \lambda I = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -3 & 5 \\ 0 & 1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{bmatrix}$$

Thus the characteristic polynomial is $\det(A - \lambda I) = (1-\lambda)(1-\lambda)(2-\lambda)$.

b. (5 points) What are the eigenvalues of A? *The eigenvalues are the roots of the characteristic polynomial. Thus $\lambda = 1$ and 2.*

c. (10 points) For each eigenvalue, find the corresponding eigenspace. Be sure to label your answer, so that it is clear which solutions go with which eigenvalues.

The eigenspace for $\lambda = 1$ is Nul $(A - 1I) =$ Nul $\begin{bmatrix} 0 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. *Thus we find a basis for the eigenspace for $\lambda = 1$ by solving the equation $(A - 1I)\mathbf{x} = \mathbf{0}$.*

$$\left[\begin{array}{ccc|c} 0 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & -5/3 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & -5/3 & 0 \\ 0 & 0 & -7/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So the solution is given by

$$\begin{aligned} &x_1 \text{ is free,} \\ &x_2 = 0 \\ &x_3 = 0 \end{aligned}$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1,$$

and the eigenspace for $\lambda = 1$ *is span* $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

The eigenspace for $\lambda = 2$ *is Nul* $(A - 2I) =$ *Nul* $\begin{bmatrix} -1 & -3 & 5 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$. *Thus we find a basis for the eigenspace for* $\lambda = 2$ *by solving the equation* $(A - 2I)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} -1 & -3 & 5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solution is given by

$$\begin{aligned} &x_1 = 5x_3 \\ &x_2 = 0 \\ &x_3 \text{ is free.} \end{aligned}$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} x_3,$$

and the eigenspace for $\lambda = 2$ *is span* $\left\{ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \right\}$.

5. a. (7 points) Explain why 0 is not an eigenvalue of an invertible $n \times n$ matrix A.

   If A *is invertible then the equation* $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution, so the equation* $(A - 0I)\mathbf{x} = \mathbf{0}$ *has only the trivial solution (so no nonzero vector* $\mathbf{x}$ *satisfies the equation). This means 0 is not an eigenvalue of* A *since any eigenvalue* λ *of* A *must have a nonzero eigenvector* $\mathbf{x}$ *that satisfies* $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

   b. (7 points) Can a 4×10 matrix A have a null space of dimension 2? Why or why not?

   A 4×10 *matrix A has 10 columns. If the null space of A is dimension 2 then A has two non-pivot columns. This means that it must have 8 pivot columns. But since A has only 4 rows, this is impossible. Thus the null space cannot have dimension 2.*

   c. (7 points) Explain why if a matrix A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ (i.e. explain why $A\mathbf{x} = \mathbf{b}$ cannot be inconsistent).

   The only way for $A\mathbf{x} = \mathbf{b}$ *to be inconsistent is for the reduced echelon form of the augmented matrix* $[A|\mathbf{b}]$ *to have a row of the form* $[0 \dots 0|c]$, *where c is a nonzero scalar. Since A has a pivot position in every row, the reduced form of the coefficient matrix will not have a row of all zeros, so* *the augmented matrix* $[A|\mathbf{b}]$ *cannot have a row of the form* $[0\dots0|c]$*, where* c *is a nonzero scalar. Thus* $A\mathbf{x}=\mathbf{b}$ *is consistent for every* $\mathbf{b}$.

6. a. (10 points) Let W be the set of polynomials of the form $p(t)=2a+bt+at^2$. Is W a subspace of $\mathbb{P}_2$? Why or why not?

The polynomials in W *have the form* $p(t)=2a+bt+at^2=a(2+t^2)+bt$. *Thus* W *is the set of all linear combinations of* $2+t^2$ *and* t. *So we have*

$$W=Span\{2+t^2,t\},$$

and W *is a subspace since all spans are subspaces.*

b. (10 points) Let $H=\left\{\begin{pmatrix} s+2t \\ 3s+6t \\ -4s-8t \end{pmatrix} : s,t\in\mathbb{R}\right\}$. Is H a subspace of $\mathbb{R}^3$? Why or why not? If it is a subspace, give a basis for it.

The vectors in H *satisfy*

$$\begin{bmatrix} s+2t \\ 3s+6t \\ -4s-8t \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}s+\begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}t.$$

Thus

$$H=Span\{\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix},\begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}\},$$

which means H *is a subspace since all spans are subspaces.*

7. Examples!

a. (5 points) Give an example of a system of 2 linear equations in 3 variables that is inconsistent.

$$\begin{aligned} x_1+x_2+x_3&=1 \\ x_1+x_2+x_3&=2 \end{aligned}$$

Tip: The question asks for a system of equations. It would not be correct to write an augmented matrix of an inconsistent system; to answer the question, you must write a system of equations.

b. (5 points) Give an example of a matrix A such that the linear transformation $T(\mathbf{x})=A\mathbf{x}$ is one-to-one but not onto.

Make a matrix with pivot position in every column but having a row without a pivot position.

$$A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

c. (5 points) Give an example of a linearly dependent set of three vectors in $\mathbb{R}^4$.

We need three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, *and* $\mathbf{v}_3$ *so that* $[\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3]$ *has a column without a pivot. Take for example*

$$\mathbf{v}_1=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\quad \mathbf{v}_2=\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\quad \mathbf{v}_1=\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$