Newberger Math 247 Spring 02

Sample Exam 2

1. a. (5 points) State the definition of one-to-one.
   A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called one-to-one if for each b in $\mathbb{R}^m$ there is at most one x in $\mathbb{R}^n$ with $T(x) = b$.

   b. (5 points) State the definition of onto.
   A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called one-to-one if for each b in $\mathbb{R}^m$ there is at least one x in $\mathbb{R}^n$ with $T(x) = b$.

2. Let $T(x_1, x_2, x_3) = (3x_2 + 2x_3, 3x_1 - 4x_2)$.

   a. (8 points) Find the standard matrix for T.
   $T(1,0,0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, *and* $T(0,1,0) = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, *and* $T(0,0,1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, *so the standard matrix for T is* $A = \begin{pmatrix} 0 & 3 & 2 \\ 3 & -4 & 0 \end{pmatrix}$.

   b. (5 points) Is T one-to-one?
   T is not one-to-one since there is not a pivot in every column, so, when it is consistent, the equation $A\mathbf{x} = \mathbf{b}$ will have free variables and infinitely many solutions.

   c. (5 points) Is T onto?
   T is onto since there is a pivot in every row and the equation $A\mathbf{x} = \mathbf{b}$ will have a solution for every $\mathbf{b}$.

3. a. (8 points) If A is an invertible $n \times n$ matrix, and $\mathbf{b}$ is a vector in $\mathbb{R}^n$, can $A\mathbf{x} = \mathbf{b}$ have an infinite number of solutions? Why or why not?
   $A\mathbf{x} = \mathbf{b}$ cannot have an infinite number of solutions. Since the invertible matrix A has a pivot in every row, $A\mathbf{x} = \mathbf{b}$ will be consistent for every $\mathbf{b}$, and since A has a pivot in every column, $A\mathbf{x} = \mathbf{b}$ will have no free variables, and hence a unique solution. That solution is given by $\mathbf{x} = A^{-1}\mathbf{b}$.

   b. (8 points) Let A and B be $n \times n$ invertible matrices, with $AXA^{-1} = B$. Explain why X is invertible and calculate X^{-1} in terms of A and B.
   Since A is invertible, we have A^{-1} to work with. Multiply $AXA^{-1} = B$ on the left by A^{-1}:

$$A^{-1}(AXA^{-1}) = A^{-1}B,$$

   which implies

$$(A^{-1}A)XA^{-1} = A^{-1}B$$

so

$$I(XA^{-1}) = A^{-1}B,$$

and we get

$$XA^{-1} = A^{-1}B.$$

Now multiply on the right by A:

$$(XA^{-1})A = A^{-1}BA,$$

which implies

$$X(A^{-1}A) = A^{-1}BA,$$

so

$$XI = A^{-1}BA,$$

and we get

$$X = A^{-1}BA.$$

Now we have that X is a product of A^{-1}, B and A. Since A is invertible, so is A^{-1}. It is given that B is invertible. So we have written X as a product of invertible matrices, and since the product of invertible matrices is invertible, we get that X is invertible. Now we calculate: $X^{-1} = (A^{-1}BA)^{-1} = A^{-1}B^{-1}(A^{-1})^{-1} = A^{-1}B^{-1}A$.

4. (6 points) State the definition of a basis.

   A basis of a subspace W is a set of vectors $\{v_1, \ldots, v_n\}$ such that $W =$Span$\{v_1, \ldots, v_n\}$ and such that $\{v_1, \ldots, v_n\}$ is a linearly independent set.

5. a. (8 points) Let $A = \begin{pmatrix} 1 & -1 & -8 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{pmatrix}$. Find a basis for Nul A.

   Solve the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 1 & -1 & -8 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 4 & 8 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   so the solutions have the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} x_3.$$

   A basis for Nul A is $\left\{ \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} \right\}$.

b. (8 points) Find a basis for H, where

$$H = Span\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Find the pivot columns of the matrix $\begin{pmatrix}1 & -1 & 0\\1 & -1 & 0\\1 & 0 & 1\end{pmatrix}$.

$$\begin{pmatrix}1 & -1 & 0\\1 & -1 & 0\\1 & 0 & 1\end{pmatrix} \sim \begin{pmatrix}1 & -1 & 0\\0 & 0 & 0\\0 & 1 & 1\end{pmatrix} \sim \begin{pmatrix}1 & -1 & 0\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix},$$

So the first two columns are the pivot columns. Thus the basis is the first two columns from the original matrix:

$$\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\end{pmatrix}\right\}$$

6. (6 points) State the definition of a subspace. *H is a subspace of a vector space V if it satisfies the following:*
   i. The zero vector from V is in H.
   ii. H is closed under addition, i.e. if $\mathbf{u}$ and $\mathbf{v}$ are in H then $\mathbf{u}+\mathbf{v}$ is in H.
   iii. H is closed under scalar multiplication, i.e. if c is a scalar and $\mathbf{u}$ is in H then $c\mathbf{u}$ is in H.

7. (a) Explain why H is not subspaces of $\mathbb{R}^2$.

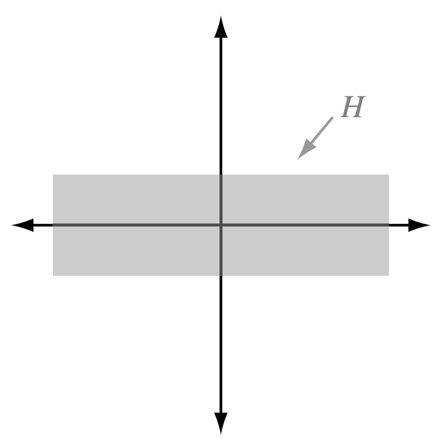
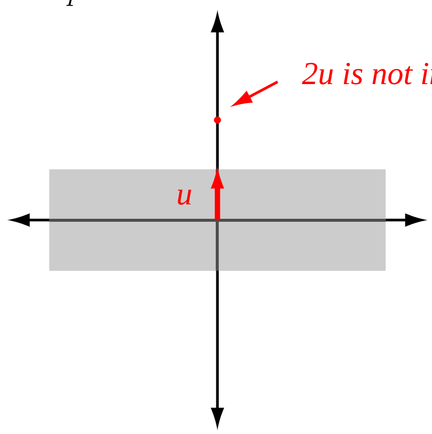


FIGURE 1. H is the grey subset of $\mathbb{R}^2$. Explain why H is not a subspace of $\mathbb{R}^2$.

I first checked that $\mathbf{0}$ *is contained in* H*. If it were not, then* H *would not be a subspace, and I would be done. Since* $\mathbf{0}$ *is in* H*, I must show that* H *is either not closed under addition or not closed under scalar multiplication. Either will work. Take for example the red vector* u *that I drew on the diagram below. If I scale it by 2, the result will not lie in* H*, so* H *is not closed under scalar multiplication.*



(b) Explain why W is not subspaces of $\mathbb{R}^2$.

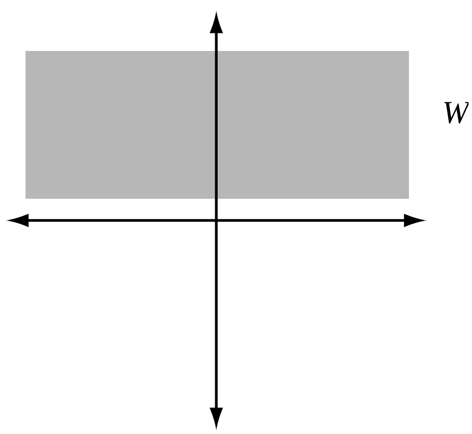


FIGURE 2. W is the dark grey subset of $\mathbb{R}^2$. Explain why W is not a subspace of $\mathbb{R}^2$.

Since $\mathbf{0}$ *is not contained in* W*,* W *is not a subspace.*

8. a. (4 points) If A is an $n \times n$ invertible matrix, what can you say about det A?

   $detA \neq 0$.

   b. (4 points) If A is an $n \times n$ invertible matrix, what can you say about Nul A?

   The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution so Nul $A = \{\mathbf{0}\}$.

   c. (4 points) If A is an $n \times n$ invertible matrix, what can you say about Col A?

   The columns of A span $\mathbb{R}^n$ so Col $A = \mathbb{R}^n$.