Newberger Math 247

Sample Exam 1 name:____________________________

1. The augmented matrix of a linear system is

$$\begin{pmatrix} 2 & -1 & -8 & 1 \\ 0 & 2 & 4 & 2 \\ 1 & 1 & -1 & 2 \end{pmatrix}.$$

a. (9 points) Reduce the augmented matrix to reduced echelon form.

$$\begin{pmatrix} 2 & -1 & -8 & 1 \\ 0 & 2 & 4 & 2 \\ 1 & 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 2 & -1 & -8 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & -6 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

b. (9 points) Write the general solution of the linear system in parametric vector form.

The solution is

$$\begin{cases} x_1 - 3x_3 = 1 \\ x_2 + 2x_3 = 1 \\ x_3 \text{ is free.} \end{cases}$$

In parametric form, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 + 3x_3 \\ 1 - 2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} x_3,$$

where x_3 *is any real number.*

c. (5 points) Give a geometric description of the solution set.

This is the line through $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ *parallel to* $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

2. Let

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}, \qquad \mathbf{a_2} = \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}.$$

a. (6 points) Is $\mathbf{b}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$? Explain.

We need to determine whether or not the system $[\mathbf{a}_1, \mathbf{a}_2]\mathbf{x} = \mathbf{b}$ *has a solution. The augmented matrix for this system is*

$$\begin{pmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \\ 9 & 10 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 \\ 0 & -4 & -12 \\ 0 & -8 & -24 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

This system is consistent, so $\mathbf{b}$ *is in Span*$\{\mathbf{a}_1, \mathbf{a}_2\}$.

b. (6 points) Do $\mathbf{a}_1$,and $\mathbf{a}_2$ span all of $\mathbb{R}^3$? Explain.

We need to determine if the matrix $[\mathbf{a}_1, \mathbf{a}_2]$ *has a pivot in every row. We see by our work in part (a) that*

$$[\mathbf{a}_1, \mathbf{a}_2] \sim \begin{pmatrix} 1 & 2 \\ 5 & 6 \\ 9 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so there is a row without a pivot, and $\mathbf{a}_1$*,and* $\mathbf{a}_2$ *do not span all of* $\mathbb{R}^3$*. We can also see this since this matrix has only 2 columns, so it cannot have 3 pivots. Thus it does not have a pivot in each of its three rows.*

c. (6 points) Is the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}\}$ linearly independent?

We need to determine whether the system $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}]\mathbf{x} = \mathbf{0}$ *has only the trivial solution or an infinite number of solutions. The coefficient matrix for this part is the same as the augmented matrix in part (a), so the system reduces*

$$\begin{pmatrix} 1 & 2 & 4 & 0 \\ 5 & 6 & 8 & 0 \\ 9 & 10 & 12 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This has one free variable and an infinite number of solutions. Therefore the the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}\}$ *is linearly dependent.*

3. Let

$$A = \begin{pmatrix} 1 & 1 & 4 \\ -3 & -3 & h \\ 1 & 8 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} -2 \\ k \\ 0 \end{pmatrix}.$$

a. (6 points) Find all values of h such that the homogeneous system $A\mathbf{x} = \mathbf{0}$ is consistent.

The homogeneous system is consistent for all values of h.

b. (6 points) Find all values of h and k such that the system $A\mathbf{x} = \mathbf{b}$ is consistent.

We reduce:

$$\begin{pmatrix} 1 & 1 & 4 & -2 \\ -3 & -3 & h & k \\ 1 & 8 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 4 & -2 \\ 0 & 0 & h+12 & k-6 \\ 0 & 7 & -4 & 2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 4 & -2 \\ 0 & 7 & -4 & 2 \\ 0 & 0 & h+12 & k-6 \end{pmatrix}$$

This will be consistent when it has a unique solution, i.e. when $h \neq -12$ and k is any real number, or when it has an infinite number of solutions, i.e. when $h = -12$ and $k = 6$.

c. (6 points) Find all values of h such that $A\mathbf{x} = \mathbf{c}$ is consistent for all possible $\mathbf{c}$.

For $A\mathbf{x} = \mathbf{c}$ to be consistent for all possible $\mathbf{c}$, there must be a pivot in every row of A, so any $h \neq -12$ will work.

d. (6 points) Is it possible for the homogeneous system $A\mathbf{x} = \mathbf{0}$ to have only the trivial solution, while $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions? Explain.

It is not possible for the system $A\mathbf{x} = \mathbf{0}$ to have only the trivial solution, while $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, since the two systems have the same number of free variables. The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution when there are no free variables, but the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions when there is at least one free variable, and these cannot happen simultaneously.

4. a. (5 points) Give an example of a system of 2 equations in 3 variables with an infinite number of solutions. (Be sure to state the equations, not just the matrix).

There must be a free variable. For example:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 2x_2 + 2x_3 = 2 \end{cases}$$

or

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 = 1 \end{cases}$$

b. (5 points) Give an example of a matrix A such that $A\mathbf{x} = \mathbf{b}$ has a solution for all possible $\mathbf{b}$.

The matrix must have a pivot in every row. For example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

c. (5 points) Give an example of two linearly independent vectors in $\mathbb{R}^3$.
The vectors must not be multiples of each other.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

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d. (5 points) Give an example of two linearly dependent vectors in $\mathbb{R}^3$.

The vectors must be multiples of each other. For example:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

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e. (5 points) Let $\mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Give an example of a vector in Span$\{\mathbf{v_1}\}$, and an example of a vector that is not in Span$\{\mathbf{v_1}\}$.
The vector in Span$\{\mathbf{v_1}\}$ *must be a multiple of* $\mathbf{v_1}$, *for example:*

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

The vector that is not in Span$\{\mathbf{v_1}\}$ *must not be a multiple of* $\mathbf{v_1}$, *for example:*

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

f. (5 points) Give an example of a matrix whose columns span all of $\mathbb{R}^2$, but are not linearly independent.

That the columns span means there must be a pivot in every row. That they are not linearly independent means that there must be a column without a pivot. For example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

g. (5 points) Give an example of a system of equations whose solution set is a line.

The system must have one free variable. For example

$$\begin{cases} x_1 + x_2 = 1, \\ x_2 \text{ is free.} \end{cases}$$